On the zero-stability of variable stepsize multistep methods: the spectral radius approach*

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Abstract

In this paper we illustrate a novel approach for studying the asymptotic behaviour of the solutions of linear difference equations with variable coefficients. In particular, we deal with the zero-stability of the 3-step BDF-method on grids with variable stepsize for the numerical solution of IVPs for ODEs. Our approach is based on the theory of the *spectral radius* of a family of matrices and yields almost optimal results, which are a sensible improvement to the best results already known from the literature. The success got on the chosen example suggests that our approach has a good potential for more general and harder stability analyses of numerical methods.

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1 Introduction

In this paper we illustrate a novel approach for studying the asymptotic behaviour of the solutions of linear difference equations with variable coefficients. In particular, we deal with the zero-stability of backward differentiation formulae (BDF-methods) for ordinary initial value problems on grids with variable stepsize.

So far, one of the most common approaches adopted in the literature consists in writing down the linear difference equation associated to the numerical method and, after choosing a convenient norm, in trying to find sufficient conditions on the coefficients of the method and on the ratioes among stepsizes in order that all the relevant companion matrices are bounded by 1 in that norm (see, e.g., Grigorieff [Gri83], Grigorieff and Paes-Leme [GPL84] and the references therein). Although very natural, such an approach has the drawback that it yields conditions which are usually much more restrictive than necessary. This is due to the fact that contractivity is required with respect to a certain norm which is chosen a priori.

On the contrary, in the approach we adopt, which is based on the theory of the *spectral radius* of a family of matrices (outlined in Section 3), the norm to use is a so called *extremal norm* and is left as an unknown of the problem to be solved for.

The method we follow to find out the extremal norm is essentially the same which, to our knowledge, has been introduced for the first time and successfully employed by Maesumi [Mae95, Mae98] for the computation of the Hölder exponent of certain wavelets.

In order to illustrate the procedure, we select the 3-step BDF-method (recalled in the next Section 2), obtaining almost optimal results, with a sensible improvement to the best known results by Grigorieff and Paes-Leme [GPL84].

Besides increasing the knowledge of this particular case, which is per se interesting, the main goal of the present paper is to promote our technique among numerical analysts and to show its potential in view of more general applications and harder stability analyses of numerical methods. However, we do not claim that our technique can always be successfully applied. In fact, as it will appear from the detailed description (see Sections 4 and 5), it is somewhat complicated and it is clear that the level of difficulty grows with the order of the difference equation to treat.

2 3-step BDF-method with variable stepsize

For the initial value problem

$$y'(t) = f(t, y(t)), \quad y(t_0) = y_0,$$

consider the grid $\Delta = \{t_0, t_1, \ldots, t_j, \ldots\}$ characterized by the stepsizes $h_j = t_{j+1} - t_j$, $j = 0, 1, \ldots$, and the 3-step BDF-method defined by

$$y_{j+3} = \alpha_{j,2} y_{j+2} + \alpha_{j,1} y_{j+1} + \alpha_{j,0} y_j + h_j \beta_j f(t_{j+3}, y_{j+3}).$$

The coefficients $\alpha_{j,s}$, $s = 0, 1, 2$, depend on the stepsize ratioes

$$\omega_{j,1} = \frac{h_{j+1}}{h_j} \quad \text{and} \quad \omega_{j,2} = \frac{h_{j+2}}{h_{j+1}}. \tag{2.1}$$

As is wellknown, the zero-stability is equivalent to the uniform boundedness of all the solutions of the associated homogeneous linear difference equation

$$u_{j+3} = \alpha_{j,2} u_{j+2} + \alpha_{j,1} u_{j+1} + \alpha_{j,0} u_j, \tag{2.2}$$

which, in turn, is equivalent to the uniform boundedness of the sequence of products

$$A^{(\nu)} \ldots A^{(0)}, \quad \nu = 0, 1, \ldots,$$

where the 3×3-matrices

$$A^{(j)} = \begin{bmatrix} \alpha_{j,2} & \alpha_{j,1} & \alpha_{j,0} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad j = 0, 1, \ldots,$$

form the sequence of companion matrices associated to the difference equation (2.2).

As it is explained in Grigorieff [Gri83] (see also Hairer, Nørsett and Wanner [HNW93]), the presence of the common eigenvector $[1, 1, 1]^T$, corresponding to the common eigenvalue $\lambda = 1$, makes it possible to reduce the dimension of one unit by using the similarity transformation $T^{-1} A^{(j)} T$ by means of the nonsingular matrix $T = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

In conclusion, the zero-stability of the BDF-method is guaranteed if

$$\lim_{\nu\to\infty} C^{(\nu)}\dots C^{(0)} = O, \tag{2.3}$$

where

$$C^{(j)} = \begin{bmatrix} \gamma_{j,1} & \gamma_{j,0} \\ 1 & 0 \end{bmatrix},$$

$$\gamma_{j,0} = -\alpha_{j,0} \quad \text{and} \quad \gamma_{j,1} = -1 + \alpha_{j,2}.$$

As one can see in [GPL84], simple but tedious calculations show that we can write the matrices $C^{(j)}$ as a function of ω_1, ω_2 as follows (here, we suppress the step index j for simplicity of notation):

$$C(\omega_1, \omega_2) = \begin{bmatrix} \gamma_1(\omega_1, \omega_2) & \gamma_0(\omega_1, \omega_2) \\ 1 & 0 \end{bmatrix}, \tag{2.4}$$

where

$$\gamma_0(\omega_1, \omega_2) = -\frac{\omega_1^3\omega_2^2(\omega_2+1)^2}{(\omega_1+1)(3\omega_1\omega_2^2+4\omega_1\omega_2+\omega_1+2\omega_2+1)} \tag{2.5}$$

and

$$\gamma_1(\omega_1, \omega_2) = \frac{\omega_2^2(\omega_1^2\omega_2^2+4\omega_1^2\omega_2+2\omega_1\omega_2+3\omega_1^2+3\omega_1+1)}{(\omega_1+1)(3\omega_1\omega_2^2+4\omega_1\omega_2+\omega_1+2\omega_2+1)}. \tag{2.6}$$

Our aim is to find the supremum Ω of the positive numbers ω such that, for any sequence of matrices $\{C^{(j)}\}_{j\geq 0}$ of the family

$$\mathcal{F}_\omega = \{C(\omega_1, \omega_2)\}_{0\leq\omega_1,\omega_2\leq\omega}, \tag{2.7}$$

it holds that (2.3) holds.

Indeed, by doing this, we risk to find more restrictive conditions on the ratioes ω_1, ω_2 than necessary. This may happen because we allow all the possible sequences of products of the elements of the family $\mathcal{F}_\omega$ to appear in (2.3), whereas it is clear that, in a sequence corresponding to the difference equation (2.2), we may only find $\omega_{j+1,1} = \omega_{j,2}$ for all j (see (2.1)).

However, our choice is motivated by the fact that, in this way, we can exploit the theory of the *spectral radius* of families of matrices (see the next Section 3) and, at the end of the computations, it turns out that, very likely, we do not loose too much with respect to the true optimal result. These are the reasons that lead us to claim that our appoach is almost optimal.

3 The spectral radius of a family of matrices

In this section we review some fundamental results regarding families of matrices, that are the key point on which we construct our technique for the stability analysis of the difference equation (2.2). For a more detailed revisitation of this topic, see also Guglielmi and Zennaro [GZ99a] and the references therein.

Let us consider a bounded family $\mathcal{F} = \{C^{(i)}\}_{i\in\mathcal{I}}$ of complex $n\times n$-matrices, where $\mathcal{I}$ is a set of indeces, possibly infinite. For such a family $\mathcal{F}$, the following definitions are given in the literature.

Let $\|\cdot\|$ be a given norm on the vector space $\mathbb{C}^n$ and let the same symbol $\|\cdot\|$ denote also the corresponding induced $n \times n$-matrix norm. Then, for each $k \geq 1$, consider the set $\Sigma_k(\mathcal{F})$ of all possible products of length k whose factors are elements of $\mathcal{F}$, that is

$$\Sigma_k(\mathcal{F}) = \{C^{(i_k)} \dots C^{(i_1)} \mid i_1, \dots, i_k \in \mathcal{I}\}.$$

Moreover, for each $k \geq 1$, consider the number

$$\hat{\rho}_k(\mathcal{F}) = \sup_{P\in\Sigma_k(\mathcal{F})} \|P\| \tag{3.1}$$

and, finally, define the *joint spectral radius* of $\mathcal{F}$ as

$$\hat{\rho}(\mathcal{F}) = \limsup_{k\to\infty} \hat{\rho}_k(\mathcal{F})^{1/k}$$

(see Rota and Strang [RS60]). Note that the numbers $\hat{\rho}_k(\mathcal{F})$ depend on the particular norm $\|\cdot\|$ used in (3.1) whereas, by the equivalence of all the norms in finite dimensional spaces, it turns out that $\hat{\rho}(\mathcal{F})$ is independent of it.

Analogously, let $\rho(\cdot)$ denote the spectral radius of an $n \times n$-matrix and then, for each $k \geq 1$, consider the number

$$\bar{\rho}_k(\mathcal{F}) = \sup_{P\in\Sigma_k(\mathcal{F})} \rho(P)$$

and define the *generalized spectral radius* of $\mathcal{F}$ as

$$\bar{\rho}(\mathcal{F}) = \limsup_{k\to\infty} \bar{\rho}_k(\mathcal{F})^{1/k}$$

(see Daubechies and Lagarias [DL92]).

It has been proved that

$$\hat{\rho}(\mathcal{F}) = \bar{\rho}(\mathcal{F})$$

(see Berger and Wang [BW92], Elsner [Els95], Shih, Wu and Pang [SWP97] and Shih [Shi99]). In the light of this equality, known as the *generalized Gelfand spectral radius formula*, we may simply talk of the *spectral radius* of the (bounded) family of matrices $\mathcal{F}$ and denote it by $\rho(\mathcal{F})$.

A first important result, strictly connected to the stability condition (2.3), is due to Berger and Wang [BW92].

Theorem 3.1 *A bounded family $\mathcal{F} = \{C^{(i)}\}_{i\in\mathcal{I}}$ of complex $n \times n$-matrices is such that*

$$\lim_{\nu\to\infty} C^{(i_\nu)} \dots C^{(i_1)} = O$$

for all sequences $\{C^{(i_j)}\}_{j\geq 1}$ of elements of $\mathcal{F}$ if and only if

$$\rho(\mathcal{F}) < 1.$$

The above definitions and results are nice generalizations of the wellknown situation for *single* families $\{C\}$. However, if the family $\mathcal{F}$ is not just a single matrix, the actual computation of $\rho(\mathcal{F})$ is not an easy task at all. Simple examples of families of just two 2×2-matrices can be given which illustrate this point (see, for example, Lagarias and Wang [LW95]).

In what follows, we shall use the notation

$$\|\mathcal{F}\| = \sup_{i\in I} \|C^{(i)}\|.$$

The following result can be found, for example, in Elsner [Els95].

Proposition 3.1 *The spectral radius of a bounded family $\mathcal{F}$ of complex $n\times n$-matrices is characterized by the equality*

$$\rho(\mathcal{F}) = \inf_{\|\cdot\|\in\mathcal{N}} \|\mathcal{F}\|, \tag{3.2}$$

where $\mathcal{N}$ denotes the set of all possible induced $n \times n$-matrix norms.

Given a family $\mathcal{F}$, an important question to answer is whether or not the inf in (3.2) is actually attained by some induced matrix norm.

Definition 3.1 *A norm $\|\cdot\|_*$ satisfying the condition*

$$\|\mathcal{F}\|_* = \rho(\mathcal{F})$$

is said to be extremal *for the family $\mathcal{F}$.*

It is wellknown that, for a single family $\{C\}$, the existence of an extremal norm is equivalent to the fact that the matrix C is *nondefective*, i.e. all of the blocks relevant to the eigenvalues of maximum modulus are diagonal in its Jordan canonical form. Another equivalent property is that, with $\hat{C} = \rho(C)^{-1}C$, the power set $\Sigma(\hat{C}) = \{\hat{C}^k \mid k \geq 1\}$ is bounded. These results generalize to a bounded family $\mathcal{F}$ as follows.

Given a bounded family $\mathcal{F} = \{C^{(i)}\}_{i\in\mathcal{I}}$ of complex $n \times n$-matrices, let us consider the *normalized* family

$$\hat{\mathcal{F}} = \rho(\mathcal{F})^{-1}\mathcal{F} = \{\rho(\mathcal{F})^{-1}C^{(i)}\}_{i\in\mathcal{I}},$$

whose spectral radius is $\rho(\hat{\mathcal{F}}) = 1$ (of course, this makes sense only if $\rho(\mathcal{F}) > 0$). Then consider the semigroup of matrices generated by $\hat{\mathcal{F}}$, i.e.

$$\Sigma(\hat{\mathcal{F}}) = \bigcup_{k\geq 1} \Sigma_k(\hat{\mathcal{F}}).$$

Definition 3.2 *A bounded family $\mathcal{F}$ of complex $n \times n$-matrices is said to be* nondefective *if the corresponding normalized family $\hat{\mathcal{F}}$ is such that the semigroup $\Sigma(\hat{\mathcal{F}})$ is a bounded set of matrices. Otherwise, if $\Sigma(\hat{\mathcal{F}})$ is unbounded, then the family $\mathcal{F}$ is said to be* defective*.*

Note that the definition of nondefective family does not involve directly the spectral properties of its elements.

The following result can be found, for example, in [RS60] and [BW92].

Proposition 3.2 *A bounded family $\mathcal{F}$ of complex $n \times n$-matrices admits an extremal norm $\|\cdot\|_*$ if and only if it is nondefective.*

In order to state another important result for our discussion, due to Elsner [Els95], we give the following definition.

Definition 3.3 *A bounded family $\mathcal{F} = \{C^{(i)}\}_{i\in\mathcal{I}}$ of complex $n \times n$-matrices is said to be* decomposable *if there exist a nonsingular $n \times n$-matrix M and two integers $n_1, n_2 \geq 1$, $n_1 + n_2 = n$, such that, for all $i \in \mathcal{I}$, it holds that*

$$M^{-1}C^{(i)}M = \begin{bmatrix} C_{11}^{(i)} & C_{12}^{(i)} \\ O & C_{22}^{(i)} \end{bmatrix},$$

where the blocks $C_{11}^{(i)}$, $C_{12}^{(i)}$, $C_{22}^{(i)}$ are $n_1 \times n_1$-, $n_1 \times n_2$- and $n_2 \times n_2$-matrices, respectively. If a family $\mathcal{F}$ is not decomposable, then it is said to be nondecomposable.

Theorem 3.2 (Elsner's Theorem) *If a bounded family $\mathcal{F}$ of complex $n \times n$-matrices is defective, then it is decomposable.*

We remark that a defective family is always decomposable, but not necessarily vice-versa. For example, for $n \geq 2$ all single families $\{C\}$ are clearly decomposable, but not necessarily defective.

Now we recall an important conjecture, arisen from work of Daubechies and Lagarias [DL92] and stated by Lagarias and Wang [LW95], whose validity would be of much help for the actual computation of the spectral radius $\rho(\mathcal{F})$ of finite families.

Definition 3.4 *If $\mathcal{F}$ is a bounded family of complex $n \times n$-matrices, any matrix $\bar{P} \in \Sigma_{k^*}(\mathcal{F})$ satisfying*

$$\rho(\mathcal{F}) = \bar{\rho}_{k^*}(\mathcal{F})^{1/k^*} = \rho(\bar{P})^{1/k^*}$$

for some integer $k^* \geq 1$ is called a spectrum-maximizing product *(in short, an* s.m.p.*) for $\mathcal{F}$.*

Conjecture 3.1 (Finiteness Conjecture) *Every finite family $\mathcal{F}$ of complex $n \times n$-matrices has an s.m.p. $\bar{P}$.*

So far, nobody has been able to disprove the Finiteness Conjecture. On the contrary, it has been partially proved by Lagarias and Wang [LW95].

Note that the finiteness of the family $\mathcal{F}$ is crucial for the possible validity of the Finiteness Conjecture. In fact, it is very easy to construct a counterexample for infinite families.

The next part of this section contains definitions and results which will be discussed and proved in a forecoming paper by Guglielmi and Zennaro [GZ99b].

Remark 3.1 *It is very easy to see that, if $\bar{P} = C^{(i_{k^*})} \ldots C^{(i_1)}$ is an s.m.p. for the family $\mathcal{F}$, then also any cyclic permutation $C^{(i_s)} \ldots C^{(i_1)} C^{(i_{k^*})} \ldots C^{(i_{s+1})}$ of $\bar{P}$, $s = 1, \ldots, k^* - 1$, is an s.m.p. for $\mathcal{F}$, along with all the powers of $\bar{P}$ and of its cyclic permutations.*

Definition 3.5 *Let $\mathcal{F}$ be a finite family of complex $n \times n$-matrices. An s.m.p. $\bar{P}$ is said to be* minimal *if it is not a power of another s.m.p. of $\mathcal{F}$.*

It is clear that, for any s.m.p. $\bar{P}$ of a family $\mathcal{F}$, it holds that either $\bar{P}$ is minimal or $\bar{P}$ is a power of another s.m.p., which is minimal.

Definition 3.6 *A finite family $\mathcal{F}$ of complex $n \times n$-matrices is said to be* asymptotically simple *if it has a unique minimal s.m.p. $\bar{P}$ (modulus cyclic permutations).*

In other words, a family $\mathcal{F}$ is asymptotically simple if there exists only one product $\bar{P}$ (modulus cyclic permutations) whose powers are asymptotically dominant.

In view of the next definition, we remark that all the cyclic permutations of a product $\bar{P}$ have the same eigenvalues with the same multiplicities.

Definition 3.7 *A finite family $\mathcal{F}$ of complex $n \times n$-matrices is said to be* absolutely asymptotically simple *if it is asymptotically simple and its minimal s.m.p. $\bar{P}$ has a unique and simple eigenvalue λ such that $|\lambda| = \rho(\bar{P})$.*

The next definitions concern the concept of *essentiality.*

Definition 3.8 *A bounded nondefective family $\mathcal{F}$ of complex $n \times n$-matrices is said to be* essential *if $\rho(\mathcal{G}) < \rho(\mathcal{F})$ for all proper subfamilies $\mathcal{G} \subset \mathcal{F}$.*

Definition 3.9 *Let $\mathcal{F}$ be a bounded family of complex $n \times n$-matrices. Every subfamily $\mathcal{G} \subseteq \mathcal{F}$ such that $\rho(\mathcal{G}) = \rho(\mathcal{F})$ is said to be* spectrum-maximizing *for $\mathcal{F}$.*

Definition 3.10 *An infinite nondefective bounded family $\mathcal{F}$ of complex $n \times n$-matrices is said to be* essentially finite *if there exists a finite spectrum-maximizing subfamily $\mathcal{G} \subset \mathcal{F}$.*

Observe that, if the Finiteness Conjecture is true, then it is automatically true also for essentially finite families.

The next discussion is devoted to the concepts of *complex polytope* and *complex polytope norm*. In this context, if $\mathcal{X}'$ and $\mathcal{X}''$ are two subsets of $\mathbb{C}^n$, we shall write $\mathcal{X}' \subset \mathcal{X}''$ ($\mathcal{X}' \supset \mathcal{X}''$) to mean that $\mathcal{X}' \subseteq \mathcal{X}''$ ($\mathcal{X}' \supseteq \mathcal{X}''$) and $\mathcal{X}' \neq \mathcal{X}''$ (*proper* inclusions).

Definition 3.11 *We say that a set $\mathcal{X} \subset \mathbb{C}^n$ is a* absolutely convex *if, for all $x', x'' \in \mathcal{X}$ and $\lambda', \lambda'' \in \mathbb{C}$ such that $|\lambda'| + |\lambda''| \leq 1$, it holds that $\lambda' x' + \lambda'' x'' \in \mathcal{X}$.*

For completeness of terminology, we recall that an absolutely convex set is *balanced* and *convex* at the same time (see, e.g., Heuser [Heu82]).

Definition 3.12 *Let $\mathcal{X} \subset \mathbb{C}^n$. Then the intersection of all absolutely convex sets containing $\mathcal{X}$ is called the* absolutely convex hull *of $\mathcal{X}$ and is denoted by* $\mathrm{absco}(\mathcal{X})$.

It is wellknown that, if $\mathcal{X} = \{x^{(i)}\}_{1 \leq i \leq m}$ is a finite set of vectors, then

$$\mathrm{absco}(\mathcal{X}) = \Big\{ x \in \mathbb{C}^n \mid x = \sum_{i=1}^{m} \lambda_i x^{(i)} \text{ with } \lambda_i \in \mathbb{C} \text{ and } \sum_{i=1}^{m} |\lambda_i| \leq 1 \Big\}.$$

The foregoing definition extends the usual definition of symmetric polytope in the real space $\mathbb{R}^n$.

Definition 3.13 *We say that a bounded set $\mathcal{P} \subset \mathbb{C}^n$ is a* balanced complex polytope *(in short, a b.c.p.) if there exists a finite set $\mathcal{X} = \{x^{(i)}\}_{1 \leq i \leq m}$ of vectors such that* $\mathrm{span}(\mathcal{X}) = \mathbb{C}^n$ *and*

$$\mathcal{P} = \mathrm{absco}(\mathcal{X}).$$

Moreover, if $\mathrm{absco}(\mathcal{X}') \subset \mathrm{absco}(\mathcal{X})$ *for all $\mathcal{X}' \subset \mathcal{X}$, then $\mathcal{X}$ is said to be an* essential system of vertices *for $\mathcal{P}$, whereas any vector $ux^{(i)}$ with $u \in \mathbb{C}$, $|u| = 1$, is said to be a* vertex *of $\mathcal{P}$.*

Remark that, geometrically speaking, a b.c.p. $\mathcal{P}$ is not a classical polytope. In fact, if we identify the complex space $\mathbb{C}^n$ with the real space $\mathbb{R}^{2n}$, we see that $\mathcal{P}$ is not bounded by hyperplanes but, rather, by quadratic surfaces.

In general, even the intersection $\mathcal{P} \bigcap \mathbb{R}^n$ is not a polytope. However, if the b.c.p. $\mathcal{P}$ admits an essential system of real vertices, then $\mathcal{P} \bigcap \mathbb{R}^n$ is a classical polytope.

The concept of *polytope norm* is extended to the complex case in a straightforward way.

Proposition 3.3 *Assume that $\mathcal{S} \subset \mathbb{C}^n$ is bounded, absolutely convex and such that* $\mathrm{span}(\mathcal{S}) = \mathbb{C}^n$*. Then it is the unit ball of a norm $\|\cdot\|_{\mathcal{S}}$ on $\mathbb{C}^n$.*

Definition 3.14 *A norm $\|\cdot\|_{\mathcal{P}}$ whose unit ball is a b.c.p. $\mathcal{P}$ is said to be a* complex polytope norm.

The second conjecture we report in this section regards the way of constructing an extremal complex polytope norm. It is a minor variant of another conjecture by Maesumi [Mae95] (see also [Mae98]) and represents a refinement of the *Extremality Conjecture* stated by Lagarias and Wang [LW95].

Before stating our conjecture, for any vector $x \in \mathbb{C}^n$ and for any normalized family $\hat{\mathcal{F}}$ we define the set

$$\mathcal{T}[\hat{\mathcal{F}}, x] = \{x\} \cup \{\hat{P}x \mid \hat{P} \in \Sigma(\hat{\mathcal{F}})\},$$

i.e. the *trajectory* obtained by applying all the normalized products $\hat{P}$ of matrices of $\hat{\mathcal{F}}$ to the vector x.

Remark that, if $\hat{\mathcal{F}}$ is nondefective, the set $\mathcal{T}[\hat{\mathcal{F}}, x]$ is bounded.

Conjecture 3.2 *Assume that a finite family $\mathcal{F}$ of complex $n \times n$-matrices is nondefective, essential and absolutely asymptotically simple. Moreover, let x be the eigenvector of the minimal s.m.p. $\bar{P}$ corresponding to the eigenvalue λ such that $|\lambda| = \rho(\bar{P})$. Then there exist a finite number of normalized products $\hat{P}^{(1)}, \ldots, \hat{P}^{(s)} \in \Sigma(\hat{\mathcal{F}})$ such that*

$$\mathrm{absco}\Big(\mathcal{T}[\hat{\mathcal{F}}, x]\Big) = \mathrm{absco}\Big(\{x, \hat{P}^{(1)}x, \ldots, \hat{P}^{(s)}x\}\Big). \tag{3.3}$$

The assumptions of Conjecture 3.2 seem to be quite restrictive. However, formula (3.3) may well hold even if some of them are not fulfilled.

If (3.3) holds and $\mathrm{span}\Big(\mathcal{T}[\hat{\mathcal{F}}, x]\Big) = \mathbb{C}^n$, the set $\mathcal{P} = \mathrm{absco}\Big(\mathcal{T}[\hat{\mathcal{F}}, x]\Big)$ is a b.c.p. and the vectors $\{x, \hat{P}^{(1)}x, \ldots, \hat{P}^{(s)}x\}$ may be chosen to form an essential system of vertices. It is immediately seen that the complex polytope norm determined by $\mathcal{P}$ is an extremal norm for the family $\mathcal{F}$. As a consequence, an extremal complex polytope norm does exist.

The following result helps in the detection of the vertices of $\mathcal{P}$.

Proposition 3.4 *Assume that a nondefective bounded family $\mathcal{F}$ of complex $n \times n$-matrices has an s.m.p. $\bar{P} = C^{(i_{k^*})} \dots C^{(i_1)}$ and let $\|\cdot\|_*$ be an extremal norm. Then, if x is an eigenvector of $\bar{P}$ corresponding to an eigenvalue λ with $|\lambda| = \rho(\bar{P})$ such that $\|x\|_* = 1$, then it holds that $\|\hat{P}x\|_* = 1$ for all normalized right factors $\hat{P} = \big(\rho(\mathcal{F})\big)^{-r} C^{(i_r)} \dots C^{(i_1)}$ of the s.m.p. $\bar{P}$, $r = 1, \dots, k^* - 1$.*

We can conclude that, among the candidates to be vertices of the b.c.p. $\mathcal{P} = \mathrm{absco}\big(\mathcal{T}[\hat{\mathcal{F}}, x]\big)$, there are at least the vectors $\hat{P}x$ for all normalized right factors $\hat{P}$ of the s.m.p. $\bar{P}$. In fact, Proposition 3.4 states that they belong to the boundary of $\mathcal{P}$, which is the unit ball of the corresponding extremal complex polytope norm.

Note that it is not important what cyclic permutation of $\bar{P}$ is chosen, since the set of vectors $\hat{P}x$ determined by the normalized right factors $\hat{P}$ is always the same, apart from a scale factor.

Finally, we remark that, if all the matrices of the family $\mathcal{F}$ are real and if also the eigenvector x is real, then Conjecture 3.2 determines a classical polytope in $\mathbb{R}^n$.

4 Computing the supremum Ω

An important preliminary observation is that, as a consequence of Theorem 3.2, the family $\mathcal{F}_\omega$ defined by (2.7), which is involved in our stability investigations, is nondefective. In fact, it can be easily seen that it is nondecomposable. Therefore, Proposition 3.2 assures the existence of an extremal norm.

The family $\mathcal{F}_\omega$ is clearly infinite. Nevertheless, we shall see that it is essentially finite and we shall find a suitable finite essential spectrum-maximizing subfamily $\mathcal{G}_\omega$.

In the light of Theorem 3.1, we can conclude that the supremum Ω of the positive numbers ω such that (2.3) holds for any sequence of matrices

$\{C^{(j)}\}_{j\geq 0}$ of $\mathcal{F}_\omega$ is nothing but

$$\Omega = \sup\{\omega > 0 \mid \rho(\mathcal{F}_\omega) < 1\}.$$

First of all, we observe that, since $\mathcal{F}_{\omega'} \subseteq \mathcal{F}_{\omega''}$ for all $0 \leq \omega' \leq \omega''$, the function $\rho(\mathcal{F}_\omega)$ is nondecreasing. That is easy, but very useful, information. In fact, since $\rho(\mathcal{F}_0) = 0$ and the function $\rho(\mathcal{F}_\omega)$ is continuous, it implies

$$\Omega = \min\{\omega > 0 \mid \rho(\mathcal{F}_\omega) = 1\}.$$

As it was already observed in [GPL84], we can claim that

$$\Omega \leq \frac{1+\sqrt{5}}{2} = 1.61803... \tag{4.1}$$

In fact, as it can be checked by means of simple but tedious calculations, the function $\rho\big(C(\omega,\omega)\big)$, which corresponds to a grid with constant stepsize ratio ω and is, of course, a lower bound to $\rho(\mathcal{F}_\omega)$, is such that

$$\rho\Big(C(\frac{1+\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2})\Big) = 1.$$

In order to compute $\rho(\mathcal{F}_\omega)$, on the basis of some preliminary numerical investigations and of the monotonicity of the functions (2.5) and (2.6) with respect to both variables ω_1, ω_2 (this can be easily checked by partial differentiation), we guess that, in a left-hand neighbourhood of $\frac{1+\sqrt{5}}{2}$, namely in $[1, \frac{1+\sqrt{5}}{2}]$, the subfamily of matrices $\mathcal{G}_\omega = \{C(0,0), C(\omega,\omega)\} \subset \mathcal{F}_\omega$ is such that $\rho(\mathcal{G}_\omega) = \rho(\mathcal{F}_\omega)$.

Moreover, on the basis of further numerical investigations (we computed $\rho(P)$ for all $P \in \Sigma_k(\mathcal{G}_\omega)$ for some low values of k), we also guess that the product

$$P(\omega) = C(0,0)C(\omega,\omega)^2 \in \Sigma_3(\mathcal{G}_\omega) \tag{4.2}$$

is a minimal s.m.p. for $\mathcal{G}_\omega$.

The truth of both our guesses would yield the equalities

$$\rho(\mathcal{F}_\omega) = \rho(\mathcal{G}_\omega) = \rho\big(P(\omega)\big)^{1/3} \tag{4.3}$$

for all $\omega \in [1, \frac{1+\sqrt{5}}{2}]$.

Numerical evidence suggests also that no minimal s.m.p. exists for $\mathcal{G}_\omega$, other than $P(\omega)$ and its cyclic permutations. Moreover, $P(\omega)$ has a unique and simple eigenvalue λ of maximum modulus (for all $\omega > 0$). Therefore, we write down a closed form for the function $\rho\big(P(\omega)\big)$ and, for all $\omega \in [1, \frac{1+\sqrt{5}}{2}]$ we define a suitable polytope $\mathcal{P}_\omega$ by using the idea suggested by Conjecture 3.2 and Proposition 3.4. Then, denoted by $\|\cdot\|_\omega$ the corresponding polytope norm, we prove that

$$\|\mathcal{F}_\omega\|_\omega = \|\mathcal{G}_\omega\|_\omega = \rho\big(P(\omega)\big)^{1/3}. \tag{4.4}$$

The final steps consist in verifying that the function $\rho\big(P(\omega)\big)$ is strictly increasing (for all $\omega > 0$) and in computing the root

$$\Omega = 1.50145... \tag{4.5}$$

of the equation $\rho\big(P(\omega)\big) = 1$.

The details of the proof are given in Section 5.

We can conclude that $\Omega = 1.50145...$ is the supremum we were looking for.

5 Details of the proof

As we anticipated in Section 4, we consider $\omega \in [1, \frac{1+\sqrt{5}}{2}]$, that is a sufficiently large left-hand neighbourhood of $\frac{1+\sqrt{5}}{2}$.

It is immediately seen that the product $P(\omega)$ in (4.2) has the form

$$P(\omega) = \begin{bmatrix} 0 & 0 \\ \Gamma_0(\omega) + \Gamma_1(\omega)^2 & \Gamma_0(\omega)\Gamma_1(\omega) \end{bmatrix},$$

where

$$\Gamma_0(\omega) = \gamma_0(\omega, \omega) = -\frac{\omega^5(\omega+1)}{3\omega^3 + 4\omega^2 + 3\omega + 1} \tag{5.1}$$

and

$$\Gamma_1(\omega) = \gamma_1(\omega, \omega) = \frac{\omega^2(\omega^3 + 3\omega^2 + 2\omega + 1)}{3\omega^3 + 4\omega^2 + 3\omega + 1}. \tag{5.2}$$

It is obvious that

$$\rho\Big(P(\omega)\Big) = -\Gamma_0(\omega)\Gamma_1(\omega). \tag{5.3}$$

All of the matrices $P(\omega)$ have the same common eigenvector

$$x^{(1)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{5.4}$$

corresponding to the eigenvalue $\Gamma_0(\omega)\Gamma_1(\omega)$ of maximum modulus.

Thus, inspired by Conjecture 3.2 and Proposition 3.4, since we are aiming at proving formula (4.3), we consider the normalized families

$$\hat{\mathcal{F}}_\omega = \rho\Big(P(\omega)\Big)^{-1/3}\mathcal{F}_\omega \quad \text{and} \quad \hat{\mathcal{G}}_\omega = \rho\Big(P(\omega)\Big)^{-1/3}\mathcal{G}_\omega$$

and define the vectors

$$x^{(i)}(\omega) = \hat{C}(\omega,\omega)^{i-1}x^{(1)}, \quad i = 2,3,4, \tag{5.5}$$

where

$$\hat{C}(\omega_1,\omega_2) = \rho\Big(P(\omega)\Big)^{-1/3}C(\omega_1,\omega_2). \tag{5.6}$$

Then we consider the polytope

$$\mathcal{Q}_\omega = \text{absco}\Big(\{x^{(1)}, x^{(2)}(\omega), x^{(3)}(\omega), x^{(4)}(\omega)\}\Big)$$

as the candidate unit ball of an extremal norm for the families $\mathcal{G}_\omega$ and $\mathcal{F}_\omega$. Therefore, our goal is reduced to prove that $\hat{\mathcal{G}}_\omega(\mathcal{Q}_\omega) \subseteq \mathcal{Q}_\omega$ and $\hat{\mathcal{F}}_\omega(\mathcal{Q}_\omega) \subseteq \mathcal{Q}_\omega$, that is

$$\hat{\mathcal{G}}_\omega(\{x^{(1)}, x^{(2)}(\omega), x^{(3)}(\omega), x^{(4)}(\omega)\}) \subseteq \mathcal{Q}_\omega \tag{5.7}$$

and

$$\hat{\mathcal{F}}_\omega(\{x^{(1)}, x^{(2)}(\omega), x^{(3)}(\omega), x^{(4)}(\omega)\}) \subseteq \mathcal{Q}_\omega. \tag{5.8}$$

Remark that, since all the matrices of $\mathcal{F}_\omega$ are real and also the eigenvector $x^{(1)}$ is real, we can limit ourselves to consider the real part of the polytope $\mathcal{Q}_\omega$ and, thus, just to work in $\mathbb{R}^2$.

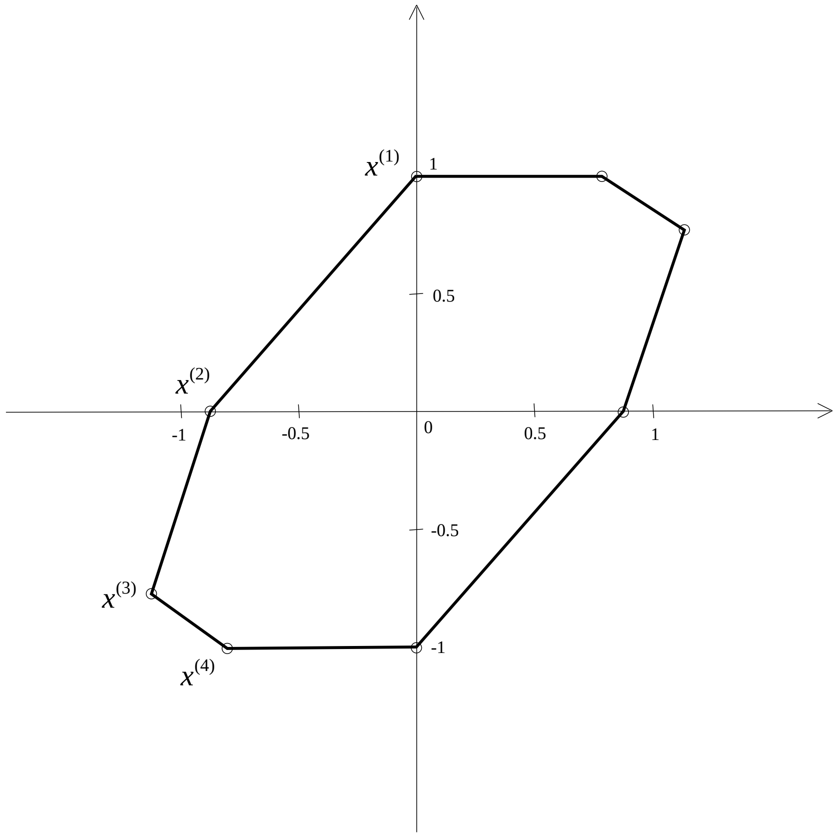

Fig. 1: Polytope $\mathcal{Q}_\omega$ for $\omega = \frac{1+\sqrt{5}}{2}$.

In order to prove (5.7) and (5.8), we observe that, besides $x^{(1)}$, by (5.4), (2.4), (5.1), (5.2), (5.6), (5.5) and (5.3), the vertices of $\mathcal{Q}_\omega$ are

$$x^{(2)} = \begin{bmatrix} (-\Gamma_0\Gamma_1)^{-1/3}\Gamma_0 \\ 0 \end{bmatrix},$$

$$x^{(3)} = \begin{bmatrix} -(-\Gamma_0\Gamma_1)^{1/3} \\ (-\Gamma_0\Gamma_1)^{-2/3}\Gamma_0 \end{bmatrix},$$

$$x^{(4)} = \begin{bmatrix} -\Gamma_1^{-1}(\Gamma_0 + \Gamma_1^2) \\ -1 \end{bmatrix},$$

where, for simplicity of notation, we omitted the dependence of ω.

Whereas it is evident that

$$x_1^{(2)} < 0, \quad x_1^{(3)} < 0, \quad x_2^{(3)} < 0,$$

some computations are needed to verify that, for $\omega \in [1, \frac{1+\sqrt{5}}{2}]$,

$$x_2^{(3)} > -1, \quad x_1^{(3)} < x_1^{(2)} < x_1^{(4)} < 0. \tag{5.9}$$

As an example, in Fig. 1 we illustrate the polytope $\mathcal{Q}_\omega$ for $\omega = \frac{1+\sqrt{5}}{2}$.

Now we apply the matrices $\hat{C}(0,0)$ and $\hat{C}(\omega,\omega)$ to the vertices of $\mathcal{Q}_\omega$ and obtain

$$\hat{C}(0,0)x^{(1)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$\hat{C}(0,0)x^{(2)} = \begin{bmatrix} 0 \\ (-\Gamma_0\Gamma_1)^{-2/3}\Gamma_0 \end{bmatrix},$$

$$\hat{C}(0,0)x^{(3)} = \begin{bmatrix} 0 \\ -1 \end{bmatrix},$$

$$\hat{C}(0,0)x^{(4)} = \begin{bmatrix} 0 \\ -(-\Gamma_0\Gamma_1)^{-1/3}\Gamma_1^{-1}(\Gamma_0 + \Gamma_1^2) \end{bmatrix},$$

$$\hat{C}(\omega,\omega)x^{(4)} = \begin{bmatrix} -(-\Gamma_0\Gamma_1)^{-1/3}(2\Gamma_0 + \Gamma_1^2) \\ -(-\Gamma_0\Gamma_1)^{-1/3}\Gamma_1^{-1}(\Gamma_0 + \Gamma_1^2) \end{bmatrix},$$

whereas $\hat{C}(\omega,\omega)x^{(i)}$, $i = 1, 2, 3$, are themselves vertices of $\mathcal{Q}_\omega$ (see (5.5)).

Some further computations reveal that, for $\omega \in [1, \frac{1+\sqrt{5}}{2}]$,

$$(\hat{C}(0,0)x^{(4)})_2 = (\hat{C}(\omega,\omega)x^{(4)})_2 > -1, \quad x_1^{(4)} < (\hat{C}(\omega,\omega)x^{(4)})_1 < 0.$$

Therefore, since $(\hat{C}(0,0)x^{(2)})_2 = x_2^{(3)}$, (5.9) allows us to conclude that (5.7) holds, i.e. the polytope norm associated to $\mathcal{Q}_\omega$ is extremal for the subfamily $\mathcal{G}_\omega$.

Now we go on with investigating the validity of (5.8). Unfortunately, we see that it does not hold. In fact, by (2.4), (2.5), (2.6) and (5.6), we obtain

$$\hat{C}(\omega_1,\omega_2)x^{(3)} = \begin{bmatrix} -\Gamma_1^{-1}(\gamma_0 + \Gamma_1\gamma_1) \\ -1 \end{bmatrix},$$

where, for simplicity of notation, we omitted also the dependence of ω_1, ω_2. Now, it turns out that, for some values of $\omega_1, \omega_2 \in [0, \omega]$, the inequality $(\hat{C}(\omega_1, \omega_2)x^{(3)})_1 < x_1^{(4)}$ holds, meaning that $\hat{C}(\omega_1, \omega_2)x^{(3)} \notin \mathcal{Q}_\omega$. On the other hand, we have

$$\hat{C}(\omega_1, \omega_2)x^{(1)} = \begin{bmatrix} (-\Gamma_0\Gamma_1)^{-1/3}\gamma_0 \\ 0 \end{bmatrix},$$

$$\hat{C}(\omega_1, \omega_2)x^{(2)} = \begin{bmatrix} (-\Gamma_0\Gamma_1)^{-2/3}\Gamma_0\gamma_1 \\ (-\Gamma_0\Gamma_1)^{-2/3}\Gamma_0 \end{bmatrix},$$

and hence, the functions (2.5) and (2.6) being monotone with respect to both variables ω_1, ω_2 (this can be easily checked by partial differentiation), it holds that

$$x_1^{(2)} < (\hat{C}(\omega_1, \omega_2)x^{(1)})_1 < 0, \quad x_1^{(3)} < (\hat{C}(\omega_1, \omega_2)x^{(2)})_1 < 0,$$

meaning that $\hat{C}(\omega_1, \omega_2)x^{(1)}, \hat{C}(\omega_1, \omega_2)x^{(2)} \in \mathcal{Q}_\omega$ for all $\omega_1, \omega_2 \in [0, \omega]$ and $\omega \in [1, \frac{1+\sqrt{5}}{2}]$.

This suggests that we can recover the situation just by substituting the vertex $x^{(4)}$ by another suitable vector, namely

$$\tilde{x}^{(4)} = \begin{bmatrix} x_1^{(3)} \\ -1 \end{bmatrix} = \begin{bmatrix} -(-\Gamma_0\Gamma_1)^{1/3} \\ -1 \end{bmatrix},$$

and, consequently, by considering the modified polytope

$$\mathcal{P}_\omega = \mathrm{absco}\Big(\{x^{(1)}, x^{(2)}, x^{(3)}, \tilde{x}^{(4)}\}\Big)$$

as the candidate unit ball of an extremal norm for the whole family $\mathcal{F}_\omega$.

Clearly, it still holds that $\hat{C}(\omega_1, \omega_2)x^{(1)}, \hat{C}(\omega_1, \omega_2)x^{(2)} \in \mathcal{P}_\omega$. Moreover, we have

$$\hat{C}(\omega_1, \omega_2)\tilde{x}^{(4)} = \begin{bmatrix} -\gamma_1 - (-\Gamma_0\Gamma_1)^{-1/3}\gamma_0 \\ -1 \end{bmatrix}.$$

Thus, some final computations show that

$$\tilde{x}_1^{(4)} < (\hat{C}(\omega_1, \omega_2)x^{(3)})_1 < (\hat{C}(\omega_1, \omega_2)\tilde{x}^{(4)})_1 < 0,$$

meaning that also $\hat{C}(\omega_1, \omega_2)x^{(3)}, \hat{C}(\omega_1, \omega_2)\tilde{x}^{(4)} \in \mathcal{P}_\omega$ for all $\omega_1, \omega_2 \in [0, \omega]$ and $\omega \in [1, \frac{1+\sqrt{5}}{2}]$.

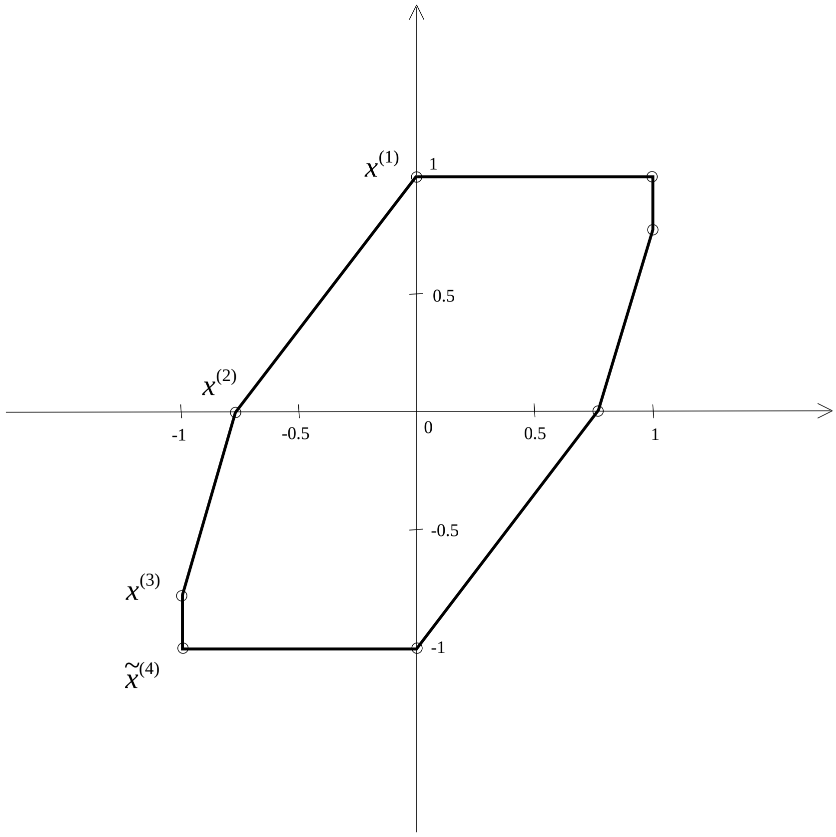

Fig. 2: Polytope $\mathcal{P}_\omega$ for $\omega = 1.50145...$.

In conclusion, we have proved that

$$\hat{\mathcal{F}}_\omega(\{x^{(1)}, x^{(2)}, x^{(3)}, \tilde{x}^{(4)}\}) \subseteq \mathcal{P}_\omega,$$

that is $\hat{\mathcal{F}}_\omega(\mathcal{P}_\omega) \subseteq \mathcal{P}_\omega$.

Therefore, (4.3) and (4.4) hold, where $\|\cdot\|_\omega$ is the polytope norm associated to $\mathcal{P}_\omega$.

Finally, by differentiation we immediately see that the function $\rho(\mathcal{F}_\omega) = \rho\Big(P(\omega)\Big)^{1/3}$ is strictly increasing for all $\omega > 0$ and, by using a root finding routine, we obtain

$$\rho\Big(P(\omega)\Big) = 1 \quad \text{if and only if} \quad \omega = \Omega = 1.50145...$$

In Fig. 2 we illustrate the polytope $\mathcal{P}_\omega$ for the value $\omega = \Omega = 1.50145...$.

6 Conclusions

The value in (4.5) valuably improves the best known result, given in [GPL84], where zero-stability is assured for $\omega \leq 1.2928...$. Moreover, in the light of the upper bound (4.1), it is very likely not far from being sharp.

The winning feature of our approach lies in the fact that we do not impose contractivity in any predefined norm. In fact, the intrinsic asymptotic properties of the products of matrices associated to a difference equation are determined by the spectral radius of the family of such matrices, which is exactly what we try to compute.

Therefore, our procedure for analyzing the asymptotic stability of linear difference equations with variable coefficients seems to have a good potential in view of much more general applications.

However, as it appears from the detailed description of our technique, the success is not guaranteed for all problems of this type. In fact, it is clear that, for each particular problem, a suitable guess has to be found for a spectrum-maximizing product of the associated family of matrices and, moreover, a suitable polytope has to be constructed as the candidate unit ball of an extremal norm.

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