On the asymptotic properties of a family of matrices*

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Abstract

In this paper we consider bounded families $\mathcal{F}$ of complex $n \times n$-matrices. After introducing the concept of *asymptotic order*, we investigate how the norm of products of matrices behaves as the number of factors goes to infinity. In the case of *defective* families $\mathcal{F}$, using the asymptotic order allows us to get a deeper knowledge of the asymptotic behaviour than just considering the so called *generalized spectral radius*. With reference to the wellknown Finiteness Conjecture for finite families, we also introduce the concepts of *spectrum-maximizing product* and *limit spectrum-maximizing product*, showing that, for finite families of 2×2-matrices, defectivity is equivalent to the existence of defective such limit products.

*This work was supported by the Italian M.U.R.S.T. and C.N.R.

1 Introduction

In this paper we consider a bounded family $\mathcal{F} = \{A^{(i)}\}_{i\in\mathcal{I}}$ of complex $n \times n$-matrices, where $\mathcal{I}$ is a set of indeces, possibly infinite. For such a family $\mathcal{F}$, the following definitions are given in the literature.

Let $\|\cdot\|$ be a given norm on the vector space $\mathbb{C}^n$ and let the same symbol $\|\cdot\|$ denote also the corresponding induced $n \times n$-matrix norm. Then, for each $k = 0, 1, \ldots$, consider the set $\Sigma_k(\mathcal{F})$ of all possible products of length k whose factors are elements of $\mathcal{F}$, that is

$$\Sigma_k(\mathcal{F}) = \{A^{(i_1)} \ldots A^{(i_k)} \mid i_1, \ldots, i_k \in \mathcal{I}\},$$

with the convention that $\Sigma_0(\mathcal{F}) = \{I\}$, I the identity matrix. Moreover, for each $k = 0, 1, \ldots$, consider the number

$$\hat{\rho}_k(\mathcal{F}) = \sup_{P\in\Sigma_k(\mathcal{F})} \|P\| \tag{1.1}$$

and, finally, define the *joint spectral radius* of $\mathcal{F}$ as

$$\hat{\rho}(\mathcal{F}) = \limsup_{k\to\infty} \hat{\rho}_k(\mathcal{F})^{1/k} \tag{1.2}$$

(see Rota and Strang [RS60]). Note that the numbers $\hat{\rho}_k(\mathcal{F})$ depend on the particular norm $\|\cdot\|$ used in (1.1) whereas, by the equivalence of all the norms in finite dimensional spaces, it turns out that $\hat{\rho}(\mathcal{F})$ is independent of it.

Analogously, let $\rho(\cdot)$ denote the spectral radius of an $n \times n$-matrix and then, for each $k = 0, 1, \ldots$, consider the number

$$\bar{\rho}_k(\mathcal{F}) = \sup_{P\in\Sigma_k(\mathcal{F})} \rho(P)$$

and define the *generalized spectral radius* of $\mathcal{F}$ as

$$\bar{\rho}(\mathcal{F}) = \limsup_{k\to\infty} \bar{\rho}_k(\mathcal{F})^{1/k}$$

(see Daubechies and Lagarias [DL92]).

It has been first proved (see [DL92]) that

$$\bar{\rho}_k(\mathcal{F}) \le \bar{\rho}(\mathcal{F})^k \le \hat{\rho}(\mathcal{F})^k \le \hat{\rho}_k(\mathcal{F}) \quad \text{for all } k \ge 0, \tag{1.3}$$

from which it follows that

$$\hat{\rho}(\mathcal{F}) = \lim_{k\to\infty} \hat{\rho}_k(\mathcal{F})^{1/k} \tag{1.4}$$

and

$$\bar{\rho}(\mathcal{F}) = \sup_{k\geq 1} \bar{\rho}_k(\mathcal{F})^{1/k}. \tag{1.5}$$

Later, as a much harder result to prove, it was shown that

$$\hat{\rho}(\mathcal{F}) \leq \bar{\rho}(\mathcal{F})$$

(see Berger and Wang [BW92], Elsner [Els95] and Shih, Wu and Pang [SWP97]).

In the light of the above inequality, it is concluded that the joint spectral radius and the generalized spectral radius of $\mathcal{F}$ are the same number, which we shall simply call the *spectral radius* of the (bounded) family of matrices $\mathcal{F}$ and denote by $\rho(\mathcal{F})$.

The above definitions and results are nice generalizations of the wellknown situation for *single* families. In particular, the equality $\hat{\rho}(\mathcal{F}) = \bar{\rho}(\mathcal{F})$ is the generalization of the so called Gelfand limit.

In practical applications, the actual computation of $\rho(\mathcal{F})$ is very important. In particular, it turns out that the *asymptotic convergence* property of a family $\mathcal{F}$ can be characterized by the inequality $\rho(\mathcal{F}) < 1$. It is almost superfluous to remark that the asymptotic behaviour of a family $\mathcal{F}$ is strictly connected to the stability properties of certain difference equations with variable coefficients, which are often involved in the study of many real-life phenomena.

Unfortunately, if the family $\mathcal{F}$ is not just a single matrix, the computation of $\rho(\mathcal{F})$ is not an easy task at all. Simple examples of families of just two 2×2-matrices can be given which illustrate this point (see, for example, [LW95]). However, for finite families, there exists a conjecture, arisen from work of Daubechies and Lagarias [DL92] and stated by Lagarias and Wang [LW95], whose validity would be of much help for the scope.

Conjecture 1.1 (Finiteness Conjecture) *If $\mathcal{F}$ is a finite family of complex $n \times n$-matrices, there exist $k^* \geq 1$ and a product $\bar{P} \in \Sigma_{k^*}(\mathcal{F})$ such that*

$$\rho(\mathcal{F}) = \bar{\rho}_{k^*}(\mathcal{F})^{1/k^*} = \rho(\bar{P})^{1/k^*}. \tag{1.6}$$

Indeed, Lagarias and Wang [LW95] stated Conjecture 1.1 for families of real matrices. However, by using the multiplicative group homomorphism Φ which associates the real $2n \times 2n$-matrix $\begin{bmatrix} \Re(A) & -\Im(A) \\ \Im(A) & \Re(A) \end{bmatrix}$ to the complex $n \times n$-matrix A, it is easy to see that the validity of the Finiteness Conjecture for all families of real matrices implies also its validity for all families of complex matrices. In fact, the homomorphism Φ preserves the eigenvalues with their algebraic and geometric multiplicities and, in particular, the spectral radius.

So far nobody has been able to disprove the Finiteness Conjecture. On the contrary, it has been partially proved by Lagarias and Wang [LW95].

In the present paper we concentrate mostly on the case of *defective* families $\mathcal{F}$, that is families for which the semigroup $\Sigma(\mathcal{F}')$, generated by the *normalized* family $\mathcal{F}'$, is unbounded (see Definition 2.3).

More precisely, after revisiting in Section 2 some useful results already known from the literature, in Section 3 we introduce the concept of *asymptotic order* of a bounded family of matrices $\mathcal{F}$. This concept represents a more sophisticated tool for the study of the asymptotic properties of a defective family than the spectral radius is. On the contrary, it does not give any further information in the case of *nondefective* families. Still in Section 3 we give some simple general lower and upper bounds to the asymptotic order of a defective family.

In Section 4 we reconsider the Finiteness Conjecture and give the definitions of *spectrum-maximizing product* and *order-maximizing product*. Then, on the basis of two instructive examples, we conclude that a possible (and euristically natural) stronger reformulation of the Finiteness Conjecture, which takes the asymptotic order into account, is false.

In Section 5 we define the so called *limit spectrum-maximizing products* and prove that, at least for finite families of 2×2-matrices, there is a strict connection between defectivity and existence of defective such limit products.

Finally, in Section 6 some conclusions are drawn.

2 Preliminary results from the literature

In what follows, for the bounded family $\mathcal{F} = \{A^{(i)}\}_{i \in \mathcal{I}}$ of complex $n \times n$-matrices, if $\|\cdot\|$ denotes a norm on the vector space $\mathbb{C}^n$ and the corresponding induced $n \times n$-matrix norm, we shall still use the same notation to define

$$\|\mathcal{F}\| = \hat{\rho}_1(\mathcal{F}) = \sup_{i \in I} \|A^{(i)}\|.$$

The following result can be found, for example, in Elsner [Els95].

Proposition 2.1 *The spectral radius of a bounded family $\mathcal{F}$ of complex $n \times n$-matrices is characterized by the equality*

$$\rho(\mathcal{F}) = \inf_{\|\cdot\| \in \mathcal{N}} \|\mathcal{F}\|, \tag{2.1}$$

where $\mathcal{N}$ denotes the set of all possible induced $n \times n$-matrix norms.

Proof. The bound to $\rho(\mathcal{F})$ by means of the right hand side of (2.1) is easily obtained. In fact, for any $\|\cdot\| \in \mathcal{N}$, (1.1) immediately yields $\hat{\rho}_k(\mathcal{F}) \leq \|\mathcal{F}\|^k$ and, therefore, $\rho(\mathcal{F}) = \hat{\rho}(\mathcal{F}) \leq \|\mathcal{F}\|$.

In order to prove the opposite inequality, given a norm $\|\cdot\|$ on the space of the vectors $x \in \mathbb{C}^n$, for any $\epsilon > 0$ define

$$\|x\|_{*,\epsilon} = \sup_{k \geq 0} \sup_{P \in \Sigma_k(\mathcal{F})} \frac{\|Px\|}{(\rho(\mathcal{F}) + \epsilon)^k}. \tag{2.2}$$

By (1.2), there exists $k_\epsilon \geq 0$ such that $\frac{\|P\|}{(\rho(\mathcal{F})+\epsilon)^k} \leq 1$ for all $k \geq k_\epsilon$ and for all $P \in \Sigma_k(\mathcal{F})$ and, therefore, there exists a constant $C > 0$ such that $\|x\|_{*,\epsilon} \leq C\|x\|$ for all $x \in \mathbb{C}^n$. Moreover, all the other properties which characterize a vector norm are easily verified. Then consider a matrix $A^{(i)} \in \mathcal{F}$ and a vector $x \in \mathbb{C}^n$ and compute

$$\begin{aligned}
\|A^{(i)}x\|_{*,\epsilon} &= \sup_{k \geq 0} \sup_{P \in \Sigma_k(\mathcal{F})} \frac{\|PA^{(i)}x\|}{(\rho(\mathcal{F}) + \epsilon)^k} \\
&= (\rho(\mathcal{F}) + \epsilon) \sup_{k \geq 0} \sup_{P \in \Sigma_k(\mathcal{F})} \frac{\|PA^{(i)}x\|}{(\rho(\mathcal{F}) + \epsilon)^{k+1}} \\
&\leq (\rho(\mathcal{F}) + \epsilon) \sup_{k \geq 0} \sup_{P \in \Sigma_{k+1}(\mathcal{F})} \frac{\|Px\|}{(\rho(\mathcal{F}) + \epsilon)^{k+1}} \\
&\leq (\rho(\mathcal{F}) + \epsilon)\|x\|_{*,\epsilon},
\end{aligned}$$

which implies $\|A^{(i)}\|_{*,\epsilon} \leq \rho(\mathcal{F}) + \epsilon$ and, thus,

$$\|\mathcal{F}\|_{*,\epsilon} \leq \rho(\mathcal{F}) + \epsilon, \tag{2.3}$$

which completes the proof. ■

Definition 2.1 *We shall say that a norm satisfying condition* (2.3) *is ϵ-*extremal *for the family $\mathcal{F}$.*

Given a family $\mathcal{F}$, an important question to answer is whether or not the inf in (2.1) is actually attained by some induced matrix norm $\|\cdot\|_*$. To this purpose, we give the following definition.

Definition 2.2 *We shall say that a norm satisfying the condition*

$$\|\mathcal{F}\| = \rho(\mathcal{F}) \tag{2.4}$$

is extremal *for the family $\mathcal{F}$.*

As we shall see, the question for a family $\mathcal{F}$ to admit an extremal norm or not is particularly important in connection with its asymptotic properties.

For a single family $\{A\}$ it is wellknown that the existence of an extremal norm is equivalent to the fact that the matrix A is *nondefective*, i.e. all of the blocks relevant to the eigenvalues of maximum modulus are diagonal in its Jordan canonical form. Another equivalent property is that, with $A' = \rho(A)^{-1}A$, the power set $\Sigma(A') = \{(A')^k \mid k \geq 1\}$ is bounded. These results generalize to a bounded family $\mathcal{F}$ as follows.

Given a bounded family $\mathcal{F} = \{A^{(i)}\}_{i\in\mathcal{I}}$ of complex $n \times n$-matrices, let us consider the *normalized* family

$$\mathcal{F}' = \{\rho(\mathcal{F})^{-1}A^{(i)}\}_{i\in\mathcal{I}},$$

whose spectral radius is $\rho(\mathcal{F}') = 1$. Then consider the semigroup of matrices generated by $\mathcal{F}'$, i.e.

$$\Sigma(\mathcal{F}') = \bigcup_{k\geq 1} \Sigma_k(\mathcal{F}').$$

Definition 2.3 *A bounded family $\mathcal{F}$ of complex $n \times n$-matrices is said to be* defective *if the corresponding normalized family $\mathcal{F}'$ is such that the semigroup $\Sigma(\mathcal{F}')$ is an unbounded set of matrices. Otherwise, if $\Sigma(\mathcal{F}')$ is bounded, then the family $\mathcal{F}$ is said to be* nondefective.

Note that we gave the definition of defective family without involving directly the spectral properties of its elements. Some relationships between defectivity and the structure of the eigenspaces of the elements of the family $\mathcal{F}$ and of their products will be investigated in Section 3.

The following result can be found, for example, in [BW92].

Proposition 2.2 *A bounded family $\mathcal{F}$ of complex $n \times n$-matrices admits an extremal norm $\|\cdot\|_*$ if and only if it is nondefective.*

Proof. The necessity is trivial, since the property (2.4) for the extremal norm $\|\cdot\|_*$ implies $\|\mathcal{F}'\|_* = 1$, meaning that $\Sigma(\mathcal{F}')$ is bounded.

In order to prove the sufficiency, let $\|\cdot\|$ be a given norm on the space of the vectors $x \in \mathbb{C}^n$ and define

$$\|x\|_* = \sup_{k\geq 0} \sup_{P\in\Sigma_k(\mathcal{F})} \frac{\|Px\|}{\rho(\mathcal{F})^k}. \tag{2.5}$$

Since the normalized family $\mathcal{F}'$ is bounded, it immediately follows that there exists a constant $C > 0$ such that $\|x\|_* \leq C\|x\|$ for all $x \in \mathbb{C}^n$. Moreover, all the other properties which characterize a vector norm are easily verified. Similarly as in the proof of Proposition 2.1, it can be checked that, for any matrix $A^{(i)} \in \mathcal{F}$ and any vector $x \in \mathbb{C}^n$, it holds that $\|A^{(i)}x\|_* \leq \rho(\mathcal{F})\|x\|_*$, which implies $\|A^{(i)}\|_* \leq \rho(\mathcal{F})$. Thus the proof is complete. ■

Corollary 2.1 *A bounded family $\mathcal{F}$ of complex $n\times n$-matrices is nondefective if and only if there exists an induced norm $\|\cdot\|_*$ such that*

$$\hat{\rho}_k(\mathcal{F}) = \rho(\mathcal{F})^k \quad \text{for all } k \geq 0. \tag{2.6}$$

Proof. Assume that $\mathcal{F}$ is nondefective. Then, by Proposition 2.2, there exists an extremal norm $\|\cdot\|_*$. Therefore, if we consider such an extremal norm in (1.1), we get $\hat{\rho}_k(\mathcal{F}) \leq \rho(\mathcal{F})^k$. On the other hand, (1.3) implies the opposite inequality $\rho(\mathcal{F})^k \leq \hat{\rho}_k(\mathcal{F})$, and hence (2.6) is proved.

Conversely, let (2.6) hold. Then for $k = 1$ we get $\|\mathcal{F}\|_* = \hat{\rho}_1(\mathcal{F}) = \rho(\mathcal{F})$, saying that the norm $\|\cdot\|_*$ is extremal. Hence the result follows from Proposition 2.2. ■

From Propositions 2.1 and 2.2 it turns out that each vector norm $\|\cdot\|$ canonically determines an ϵ-extremal norm for any bounded family $\mathcal{F}$ and, if $\mathcal{F}$ is nondefective, also an extremal norm.

However, it is worth remarking that, although (2.2) and (2.5) give a constructive way of finding ϵ-extremal and extremal norms, their relevance is mainly theoretical and they often are useless from a practical point of view.

We conclude this section with the following important result by Elsner [Els95].

Theorem 2.1 *If a bounded family $\mathcal{F} = \{A^{(i)}\}_{i\in\mathcal{I}}$ of complex $n\times n$-matrices is defective, then there exist a nonsingular $n\times n$-matrix M and two integers $n_1, n_2 \geq 1$, $n_1 + n_2 = n$, such that, for all $i \in \mathcal{I}$, it holds that*

$$M^{-1}A^{(i)}M = \begin{bmatrix} \hat{A}_{11}^{(i)} & \hat{A}_{12}^{(i)} \\ O & \hat{A}_{22}^{(i)} \end{bmatrix}, \tag{2.7}$$

where the blocks $\hat{A}_{11}^{(i)}$, $\hat{A}_{12}^{(i)}$ and $\hat{A}_{22}^{(i)}$ are $n_1\times n_1$-, $n_1\times n_2$- and $n_2\times n_2$-matrices, respectively.

Proof. In view of Definiton 2.3, it is sufficient to apply Lemma 4 in [Els95] to the normalized family $\mathcal{F}'$. ■

3 The asymptotic order

Let $\alpha = \{\alpha_k\}_{k\geq 0}$ be a sequence of nonnegative real numbers and let $\mathcal{S}$ be the set of all such sequences.

We shall need the so called Θ-*notation* (see, for example, Cormen, Leiserson and Rivest [CLR92]). Given $\alpha \in \mathcal{S}$, define the subset

$$\Theta(\alpha) = \{\beta \in \mathcal{S} \mid \exists\ k_\beta \geq 1 \text{ and real } c'_\beta, c''_\beta > 0\ :\ c'_\beta\alpha_k \leq \beta_k \leq c''_\beta\alpha_k\ \forall k \geq k_\beta\}.$$

It is easy to see that the subsets $\Theta(\alpha)$ are the equivalence classes of an equivalence relation in $\mathcal{S}$, and we shall call them the *asymptotic orders* of $\mathcal{S}$. This is motivated by the fact that two sequences belong to the same class if and only if they have the same qualitative behaviour for $k \to \infty$.

For the sake of convenience, we shall denote $\Theta(\alpha)$ also by $\Theta(\alpha_k)$, so as to make explicit the particular dependence on k (for example, $\Theta(k)$, $\Theta(k^2)$, etc.).

Then consider also the so called O-*notation*, Ω-*notation* and ω-*notation* (see again [CLR92]). Given $\alpha \in \mathcal{S}$, define the subsets

$$O(\alpha) = \{\beta \in \mathcal{S} \mid \exists\ k_\beta \geq 1 \text{ and real } c''_\beta > 0\ :\ \beta_k \leq c''_\beta\alpha_k\ \forall k \geq k_\beta\},$$

$$\Omega(\alpha) = \{\beta \in \mathcal{S} \mid \exists\ k_\beta \geq 1 \text{ and real } c'_\beta > 0\ :\ c'_\beta\alpha_k \leq \beta_k\ \forall k \geq k_\beta\}$$

and

$$\omega(\alpha) = \{\beta \in \mathcal{S} \mid \forall c' > 0\ \exists\ k_{\beta,c'} \geq 1 :\ c'\alpha_k \leq \beta_k\ \forall k \geq k_{\beta,c'}\}.$$

Remark that

$$\Theta(\alpha) = O(\alpha) \bigcap \Omega(\alpha)$$

and that, if $\alpha \in \mathcal{S}$ is definitively positive,

$$\beta \in \omega(\alpha) \quad \text{if and only if} \quad \lim_{k\to\infty} \beta_k/\alpha_k = +\infty. \tag{3.1}$$

With reference to (1.1), consider the sequence $\hat{\rho}_{\mathcal{F}} = \{\hat{\rho}_k(\mathcal{F})\}_{k\geq 0}$ and, for all $k \geq 0$, define the numbers

$$\hat{\mu}_k(\mathcal{F}) = \max_{0\leq j\leq k} \rho(\mathcal{F})^{k-j} \hat{\rho}_j(\mathcal{F}) \tag{3.2}$$

and the sequence $\hat{\mu}_{\mathcal{F}} = \{\hat{\mu}_k(\mathcal{F})\}_{k\geq 0}$.

Remark that, by the equivalence of all the norms in finite dimensional spaces, both the asymptotic orders $\Theta(\hat{\rho}_{\mathcal{F}}) = \Theta(\hat{\rho}_k(\mathcal{F}))$ and $\Theta(\hat{\mu}_{\mathcal{F}}) = \Theta(\hat{\mu}_k(\mathcal{F}))$ are independent of the particular norm $\|\cdot\|$ used in (1.1).

For $j = k$, formula (3.2) clearly yields

$$\hat{\mu}_k(\mathcal{F}) \geq \hat{\rho}_k(\mathcal{F}) \tag{3.3}$$

and hence $\hat{\mu}_{\mathcal{F}} \in \Omega(\hat{\rho}_{\mathcal{F}})$. The opposite relashionship $\hat{\mu}_{\mathcal{F}} \in O(\hat{\rho}_{\mathcal{F}})$ has not been proved in the general case. Nevertheless, now we shall see that it holds for some important classes of families of matrices.

In fact, if $\mathcal{F}$ is a bounded nondefective family, Corollary 2.1 immediately implies

$$\hat{\mu}_{\mathcal{F}} \in \Theta(\hat{\rho}_{\mathcal{F}}). \tag{3.4}$$

Moreover, if a bounded family $\mathcal{F}$ contains the matrix $\rho(\mathcal{F})I$, the sequence $\{\rho(\mathcal{F})^{-k}\hat{\rho}_k(\mathcal{F})\}_{k\geq 0}$ is non decreasing, implying $\hat{\mu}_k(\mathcal{F}) = \hat{\rho}_k(\mathcal{F})$. Therefore, (3.4) holds also in this case.

Finally, consider a single family $\{A\}$. It is wellknown that, for any induced norm $\|\cdot\|$,

$$\hat{\rho}_k(A) = \|A^k\| \in \Theta(k^{d_A}\rho(A)^k), \tag{3.5}$$

where d_A, which we call *index of defectivity* of A, is a nonnegative integer $\leq n-1$ determined by the structure of the blocks relevant to the eigenvalues of maximum modulus in its Jordan canonical form. In particular, $d_A = 0$ for nondefective matrices. In any case, (3.5) yields $\hat{\mu}_k(A) \in \Theta(k^{d_A}\rho(A)^k)$ too, and thus

$$\hat{\mu}_A \in \Theta(\hat{\rho}_A). \tag{3.6}$$

In any case, even if the family $\mathcal{F}$ does not satisfy (3.4), it is immediately seen that

$$\rho(\mathcal{F}) = \lim_{k\to\infty} \hat{\mu}_k(\mathcal{F})^{1/k} \tag{3.7}$$

still holds (see (1.4)).

The sequence $\hat{\mu}_{\mathcal{F}}$ is nicer than $\hat{\rho}_{\mathcal{F}}$ because, whereas the sole inequality $\hat{\rho}_k(\mathcal{F}) \leq \|\mathcal{F}\| \cdot \hat{\rho}_{k-1}(\mathcal{F})$ generally holds for $\hat{\rho}_{\mathcal{F}}$, it is straightforward to prove that

$$\rho(\mathcal{F}) \cdot \hat{\mu}_{k-1}(\mathcal{F}) \leq \hat{\mu}_k(\mathcal{F}) \leq \|\mathcal{F}\| \cdot \hat{\mu}_{k-1}(\mathcal{F}) \tag{3.8}$$

for all $k \geq 1$. Consequently, the sequence $\{\rho(\mathcal{F})^{-k}\hat{\mu}_k(\mathcal{F})\}_{k\geq 0}$ is non decreasing, which, in general, is not guaranteed for the sequence $\{\rho(\mathcal{F})^{-k}\hat{\rho}_k(\mathcal{F})\}_{k\geq 0}$.

Definition 3.1 *Given a bounded family $\mathcal{F}$ of complex $n \times n$-matrices, we shall say that*

$$\operatorname{ord}(\mathcal{F}) = \Theta(\hat{\mu}_{\mathcal{F}})$$

is the asymptotic order *of $\mathcal{F}$.*

In particular, we have the asymptotic order of a single matrix A, i.e. $\operatorname{ord}(A) = \Theta(k^{d_A}\rho(A)^k)$ (see (3.6) and (3.5)).

In view of the discussion before Definition 3.1, we can reformulate Corollary 2.1 as follows.

Corollary 3.1 *A bounded family $\mathcal{F}$ of complex $n\times n$-matrices is nondefective if and only if*

$$\operatorname{ord}(\mathcal{F}) = \Theta(\rho(\mathcal{F})^k). \tag{3.9}$$

For a general defective bounded family of matrices $\mathcal{F}$, it is not possible to obtain a formula similar to (3.5) in the same easy way. Definition 3.1 is a refinement of the concept of joint spectral radius which will help us to investigate more deeply the asymptotic properties of a defective family of matrices. This will appear more evident from the forecoming results.

Proposition 3.1 *For any bounded family $\mathcal{F}$ of complex $n \times n$-matrices it holds that*

$$\operatorname{ord}(\mathcal{F}) \subseteq \bigcap_{j\geq 1} \bigcap_{P\in\Sigma_j(\mathcal{F})} \Omega(k^{d_P}\rho(P)^{k/j}),$$

where d_P is the index of defectivity of P, being $d_P = 0$ if P is nondefective.

Proof. Let $j \geq 1$ and $P \in \Sigma_j(\mathcal{F})$. If $\rho(P) = 0$, the inequality $\hat{\rho}_k(\mathcal{F}) \geq k^{d_P}\rho(P)^{k/j}$ is obvious. Thus we assume $\rho(P) > 0$.

For each $q = 1, \ldots, j$, there exist two matrix products $Q \in \Sigma_q(\mathcal{F})$ and $R \in \Sigma_{j-q}(\mathcal{F})$ such that $P = QR$. Therefore, given an induced matrix norm $\|\cdot\|$, for each $s \geq 1$ it holds that

$$\|P^s\| = \|P^{s-1}QR\| \leq \|P^{s-1}Q\| \cdot \|R\| \leq \|\mathcal{F}\|^{j-q} \cdot \|P^{s-1}Q\|,$$

and hence

$$\|P^{s-1}Q\| \geq \|\mathcal{F}\|^{q-j} \cdot \|P^s\|.$$

On the other hand, (3.5) implies that there exist $s_P \geq 1$ and $C_P > 0$ such that

$$\|P^s\| \geq C_P s^{d_P}\rho(P)^s$$

for all $s \geq s_P$ and, therefore, we obtain

$$\|P^{s-1}Q\| \geq C_P\|\mathcal{F}\|^{q-j}s^{d_P}\rho(P)^s. \tag{3.10}$$

In conclusion, for any $k = (s-1)j + q \geq (s_P - 1)j + q$, (1.1) and (3.10) yield

$$\hat{\mu}_k(\mathcal{F}) \geq \hat{\rho}_k(\mathcal{F}) \geq C_P\|\mathcal{F}\|^{q-j}\rho(P)^{1-q/j}\left(\frac{k+j-q}{j}\right)^{d_P}\rho(P)^{k/j} \tag{3.11}$$

and the result follows since, being $\rho(P) > 0$, there exists $\mu_P > 0$ such that

$$\|\mathcal{F}\|^{q-j}\rho(P)^{1-q/j}\left(\frac{k+j-q}{j}\right)^{d_P} \geq \mu_P k^{d_P}$$

for all $q = 1, \ldots, j$ and for all sufficiently large k. ■

The following result is wellknown (see Berger and Wang [BW92]).

Lemma 3.1 *Assume that, for the bounded family $\mathcal{F} = \{A^{(i)}\}_{i\in\mathcal{I}}$ of complex $n \times n$-matrices, there exists a nonsingular $n \times n$-matrix M such that, for all $i \in \mathcal{I}$,* (2.7) *holds for suitable $n_1, n_2 \geq 1$, $n_1 + n_2 = n$. Then, with $\mathcal{F}_1 = \{\hat{A}_{11}^{(i)}\}_{i\in\mathcal{I}}$ and $\mathcal{F}_2 = \{\hat{A}_{22}^{(i)}\}_{i\in\mathcal{I}}$, it holds that*

$$\rho(\mathcal{F}) = \max\{\rho(\mathcal{F}_1), \rho(\mathcal{F}_2)\}. \tag{3.12}$$

We recall that, given two sequences $\alpha, \beta \in \mathcal{S}$, their *convolution product* is the sequence

$$\alpha * \beta = \{\sum_{j=0}^{k} \alpha_j \beta_{k-j}\}_{k\geq 0}.$$

Remark that the asymptotic order $\Theta(\alpha * \beta)$ is independent of the particular representatives for the asymptotic orders $\Theta(\alpha)$ and $\Theta(\beta)$. In fact, it is easily seen that

$$\Theta(\alpha) * \Theta(\beta) \subseteq \Theta(\alpha * \beta).$$

Now we are in a position to state some results on the asymptotic orders of the families $\mathcal{F}$, $\mathcal{F}_1$ and $\mathcal{F}_2$ in Lemma 3.1.

Lemma 3.2 *Assume the hypotheses of Lemma* 3.1*. Then*

$$\mathrm{ord}(\mathcal{F}) \subseteq \Omega(\hat{\mu}_{\mathcal{F}_1}) \bigcap \Omega(\hat{\mu}_{\mathcal{F}_2}) \bigcap O(\hat{\mu}_{\mathcal{F}_1} * \hat{\mu}_{\mathcal{F}_2}). \tag{3.13}$$

Proof. Consider two vector norms $\|\cdot\|_1$ on $\mathbb{C}^{n_1}$ and $\|\cdot\|_2$ on $\mathbb{C}^{n_2}$ and define the following norm on $\mathbb{C}^n$:

$$\|x\| = \max\{\|x_1\|_1, \|x_2\|_2\},$$

for all $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, where $x_1 \in \mathbb{C}^{n_1}$ and $x_2 \in \mathbb{C}^{n_2}$. Moreover, let $\|\cdot\|_{21}$ denote the induced matrix norm from $\mathbb{C}^{n_2}$ into $\mathbb{C}^{n_1}$.

It is straighforward to verify that, for any block upper triangular $n \times n$-matrix $U = \begin{bmatrix} U_{11} & U_{12} \\ O & U_{22} \end{bmatrix}$, the induced matrix norm $\|\cdot\|$ satisfies the bounds

$$\max\{\|U_{11}\|_1, \|U_{22}\|_2\} \leq \|U\| \leq \max\{\|U_{11}\|_1 + \|U_{12}\|_{21}, \|U_{22}\|_2\}. \tag{3.14}$$

Now consider $k \geq 1$ and $P^{(k)} \in \Sigma_k(\mathcal{F})$. Since

$$M^{-1}P^{(k)}M = \begin{bmatrix} \hat{P}_{11}^{(k)} & \hat{P}_{12}^{(k)} \\ O & \hat{P}_{22}^{(k)} \end{bmatrix}$$

and since by varying $P^{(k)}$ in $\Sigma_k(\mathcal{F})$ the diagonal blocks $\hat{P}_{11}^{(k)}$ and $\hat{P}_{22}^{(k)}$ cover completely the product sets $\Sigma_k(\mathcal{F}_1)$ and $\Sigma_k(\mathcal{F}_2)$, by (3.14) and by using the norm $\|\cdot\|_M$ defined on $\mathbb{C}^n$ by $\|x\|_M = \|Mx\|$ for all $x \in \mathbb{C}^n$, we can conclude that $\hat{\rho}_k(\mathcal{F}) \geq \hat{\rho}_k(\mathcal{F}_s)$, $s = 1,2$, for all $k \geq 0$. Therefore, by (3.2) and Lemma 3.1, we easily obtain

$$\hat{\mu}_k(\mathcal{F}) \geq \hat{\mu}_k(\mathcal{F}_s), \quad s = 1,2. \tag{3.15}$$

In order to conclude the proof, let again $k \geq 1$ and, for each $j = 0,1,\ldots,k$, consider $P^{(j)} \in \Sigma_j(\mathcal{F})$ subject to the sole condition that $P^{(j)} = P^{(j-1)}A^{(i_j)}$ for some $A^{(i_j)} \in \mathcal{F}$. Clearly, $P^{(0)} = I$, whereas $P^{(k)}$ may be chosen arbitrarily. Then, for $j \geq 1$ we have

$$\begin{aligned} M^{-1}P^{(j)}M &= \begin{bmatrix} \hat{P}_{11}^{(j-1)} & \hat{P}_{12}^{(j-1)} \\ O & \hat{P}_{22}^{(j-1)} \end{bmatrix} \cdot \begin{bmatrix} \hat{A}_{11}^{(i_j)} & \hat{A}_{12}^{(i_j)} \\ O & \hat{A}_{22}^{(i_j)} \end{bmatrix} \\ &= \begin{bmatrix} \hat{P}_{11}^{(j-1)}\hat{A}_{11}^{(i_j)} & \hat{P}_{11}^{(j-1)}\hat{A}_{12}^{(i_j)} + \hat{P}_{12}^{(j-1)}\hat{A}_{22}^{(i_j)} \\ O & \hat{P}_{22}^{(j-1)}\hat{A}_{22}^{(i_j)} \end{bmatrix}. \end{aligned}$$

Therefore, with $\hat{Q}_{22}^{(k)} = I$ and $\hat{Q}_{22}^{(j)} = \prod_{h=j+1}^{k} \hat{A}_{22}^{(i_h)} \in \Sigma_{k-j}(\mathcal{F})$, $j = 1,\ldots,k-1$, by using induction it is easy to see that

$$\hat{P}_{12}^{(k)} = \sum_{j=1}^{k} \hat{P}_{11}^{(j-1)}\hat{A}_{12}^{(i_j)}\hat{Q}_{22}^{(j-1)}.$$

Consequently, by the boundedness of the family $\mathcal{F}$ and by (3.3), there exists $C > 0$ independent of k such that

$$\begin{aligned} \|\hat{P}_{12}^{(k)}\|_{21} &\leq \sum_{j=1}^{k} \|\hat{P}_{11}^{(j-1)}\|_1 \cdot \|\hat{A}_{12}^{(i_j)}\|_{21} \cdot \|\hat{Q}_{22}^{(j-1)}\|_2 \\ &\leq C \sum_{j=0}^{k-1} \hat{\rho}_j(\mathcal{F}_1)\hat{\rho}_{k-1-j}(\mathcal{F}_2) \\ &\leq C \sum_{j=0}^{k-1} \hat{\mu}_j(\mathcal{F}_1)\hat{\mu}_{k-1-j}(\mathcal{F}_2). \end{aligned}$$

Since the convolution product is commutative, by Lemma 3.1 we can assume, without any restriction, that $\rho(\mathcal{F}) = \rho(\mathcal{F}_2)$. Then the left hand inequality in (3.8) implies

$$\sum_{j=0}^{k-1} \hat{\mu}_j(\mathcal{F}_1)\hat{\mu}_{k-1-j}(\mathcal{F}_2) \le \rho(\mathcal{F})^{-1} \sum_{j=0}^{k} \hat{\mu}_j(\mathcal{F}_1)\hat{\mu}_{k-j}(\mathcal{F}_2). \tag{3.16}$$

Therefore, by (3.14) there exists a constant $D > 0$ independent of k such that

$$\|M^{-1}P^{(k)}M\| \le D \sum_{j=0}^{k} \hat{\mu}_j(\mathcal{F}_1)\hat{\mu}_{k-j}(\mathcal{F}_2)$$

and thus, by the arbitrariness of $P^{(k)} \in \Sigma_k(\mathcal{F})$, by using again the norm $\|\cdot\|_M$ on $\mathbb{C}^n$ we obtain

$$\hat{\rho}_k(\mathcal{F}) \le D \sum_{j=0}^{k} \hat{\mu}_j(\mathcal{F}_1)\hat{\mu}_{k-j}(\mathcal{F}_2).$$

Finally, by (3.16), the sequence $\{\rho(\mathcal{F})^{-k} \sum_{j=0}^{k} \hat{\mu}_j(\mathcal{F}_1)\hat{\mu}_{k-j}(\mathcal{F}_2)\}_{k\ge 0}$ is non decreasing. So we can conclude that

$$\hat{\mu}_k(\mathcal{F}) \le D \sum_{j=0}^{k} \hat{\mu}_j(\mathcal{F}_1)\hat{\mu}_{k-j}(\mathcal{F}_2).$$

Therefore, the above inequality and (3.15) yield (3.13). ■

Proposition 3.2 *Assume the hypotheses of Lemma* 3.1 *and that* $\rho(\mathcal{F}_1) = \rho(\mathcal{F}_2) = \rho(\mathcal{F})$. *If* $d_1, d_2 \ge 0$ *are real numbers such that*

$$\operatorname{ord}(\mathcal{F}_s) \subseteq O(k^{d_s}\rho(\mathcal{F})^k), \quad s = 1, 2, \tag{3.17}$$

then

$$\operatorname{ord}(\mathcal{F}) \subseteq O(k^{d_1+d_2+1}\rho(\mathcal{F})^k). \tag{3.18}$$

Proof. Since $k^{d_s}\rho(\mathcal{F})^k > 0$, $s = 1, 2$, the inclusions (3.17) provide the existence of two constants $C_1, C_2 > 0$ such that $\hat{\mu}_k(\mathcal{F}_s) \leq C_s k^{d_s}\rho(\mathcal{F})^k$ for all $k \geq 0$, $s = 1, 2$. Therefore it follows that

$$\sum_{j=0}^{k} \hat{\mu}_j(\mathcal{F}_1)\hat{\mu}_{k-j}(\mathcal{F}_2) \leq C_1C_2\rho(\mathcal{F})^k \sum_{j=0}^{k} j^{d_1}(k-j)^{d_2} \leq C_1C_2k^{d_1+d_2+1}\rho(\mathcal{F})^k,$$

that is (3.18). ■

Proposition 3.3 *Assume the hypotheses of Lemma* 3.1 *and that* $\rho(\mathcal{F}_1) \neq \rho(\mathcal{F}_2)$. *Then*

$$\operatorname{ord}(\mathcal{F}) = \operatorname{ord}(\mathcal{F}_s), \tag{3.19}$$

where s *is such that* $\rho(\mathcal{F}_s) = \rho(\mathcal{F})$.

Moreover, the family $\mathcal{F}$ *is nondefective if and only if* $\mathcal{F}_s$ *is nondefective.*

Proof. The key-point of the proof is the use of (3.13) and thus, since the convolution product is commutative, we can assume, without any restriction, that $s = 1$.

Then, let $\epsilon > 0$ such that $r = \frac{\rho(\mathcal{F}_2)+\epsilon}{\rho(\mathcal{F}_1)} < 1$. Moreover, consider an ϵ-extremal norm for the family $\mathcal{F}_2$. By the right-hand side inequality in (3.8) and by the monotonicity of the sequence $\{\rho(\mathcal{F}_1)^{-j}\hat{\mu}_j(\mathcal{F}_1)\}_{j\geq 0}$, we obtain

$$\sum_{j=0}^{k} \hat{\mu}_j(\mathcal{F}_1)\hat{\mu}_{k-j}(\mathcal{F}_2) \leq \sum_{j=0}^{k} \rho(\mathcal{F}_1)^{k-j}\hat{\mu}_j(\mathcal{F}_1)r^{k-j} \leq \frac{\hat{\mu}_k(\mathcal{F}_1)}{1-r},$$

which implies $\hat{\mu}_{\mathcal{F}_1} * \hat{\mu}_{\mathcal{F}_2} \in O(\hat{\mu}_{\mathcal{F}_1})$. Therefore, by (3.13), we get (3.19).

Finally, since $\rho(\mathcal{F}_s) = \rho(\mathcal{F})$, Corollary 3.1 immediately implies that $\mathcal{F}$ is nondefective if and only if $\mathcal{F}_s$ is nondefective. ■

Theorem 3.1 *A bounded family* $\mathcal{F}$ *of complex* $n \times n$*-matrices is defective if and only if*

$$\operatorname{ord}(\mathcal{F}) \subseteq \omega(\rho(\mathcal{F})^k) \bigcap O(k^{n-1}\rho(\mathcal{F})^k). \tag{3.20}$$

Proof. The sufficiency is obvious because of Corollary 3.1.

In order to prove the necessity, we start by observing that, if the family $\mathcal{F} = \{A^{(i)}\}_{i\in\mathcal{I}}$ is defective, then $\mathrm{ord}(\mathcal{F}) \subseteq \Omega(\rho(\mathcal{F})^k)$.

Since the sequence $\{\rho(\mathcal{F})^{-k}\hat{\mu}_k(\mathcal{F})\}_{k\geq 0}$ is non decreasing, it is either bounded or diverging. If it were bounded we should have $\mathrm{ord}(\mathcal{F}) \subseteq O(\rho(\mathcal{F})^k)$ and, hence, $\mathrm{ord}(\mathcal{F}) = \Theta(\rho(\mathcal{F})^k)$. On the other hand, this is impossible because of Corollary 3.1. Therefore, the sequence $\{\rho(\mathcal{F})^{-k}\hat{\mu}_k(\mathcal{F})\}_{k\geq 0}$ is diverging, which means $\mathrm{ord}(\mathcal{F}) \subseteq \omega(\rho(\mathcal{F})^k)$ (see (3.1)).

We complete the proof by proving inductively the inclusion

$$\mathrm{ord}(\mathcal{F}) \subseteq O(k^{n-1}\rho(\mathcal{F})^k) \tag{3.21}$$

also for nondefective families (doing this allows us to start the induction from $n = 1$, in which case the matrices are just numbers and the family always is nondefective).

It is clear that (3.21) holds for $n = 1$, in which case $\hat{\mu}_k(\mathcal{F}) = \rho(\mathcal{F})^k = \left(\sup_{i\in\mathcal{I}} |A^{(i)}|\right)^k$.

Now let $\bar{n} \geq 1$. Assume, by induction, that (3.21) holds for all $n \leq \bar{n} - 1$. Since the family $\mathcal{F}$ of $\bar{n}\times\bar{n}$-matrices is defective, we can apply Theorem 2.1. Therefore, since $n_1, n_2 \leq \bar{n} - 1$, with $\mathcal{F}_1 = \{\hat{A}_{11}^{(i)}\}_{i\in\mathcal{I}}$ and $\mathcal{F}_2 = \{\hat{A}_{22}^{(i)}\}_{i\in\mathcal{I}}$, we get (3.17) with $d_s = n_s - 1$, $s = 1, 2$.

If $\rho(\mathcal{F}_1) \neq \rho(\mathcal{F}_2)$, then Proposition 3.3 provides (3.19) and, a fortiori, (3.21) for $n = \bar{n}$. Otherwise, if $\rho(\mathcal{F}_1) = \rho(\mathcal{F}_2)$, Proposition 3.2 provides (3.18) and hence, being $n_1 + n_2 = \bar{n}$, again (3.21) is proved for $n = \bar{n}$, too. ■

The foregoing theorem assures, for a general defective bounded family of matrices $\mathcal{F}$, the same upper bound to the asymptotic order as in the case of a single family $\{A\}$ (see (3.5), where $d_A \leq n - 1$). Anyway, the difference with that simple case is that, in general, an exact estimate of $\mathrm{ord}(\mathcal{F})$ can not be easily obtained and, above all, is not always of the type $\Theta(k^d\rho(\mathcal{F})^k)$, d positive integer. This fact is illustrated by the following example, where d takes non integer values.

Example 3.1 Let $p > 1$ be a real number and let $\mathcal{F} = \{A^{(\theta)}\}_{\theta\in[0,1]}$ be the family of the 2×2-matrices

$$A^{(\theta)} = \begin{bmatrix} 1 & \theta \\ 0 & 1-\theta^p \end{bmatrix}.$$

This family is normalized (i.e., $\rho(\mathcal{F}) = 1$) and contains the identity matrix $I = A^{(0)}$. Therefore, $\hat{\mu}_k(\mathcal{F}) = \hat{\rho}_k(\mathcal{F})$ and the sequence $\{\hat{\mu}_k(\mathcal{F})\}_{k\geq 0}$ is non decreasing.

It is immediately seen that, for each $k \geq 2$, the products $P_k \in \Sigma_k(\mathcal{F})$ have the form

$$P_k = \prod_{j=1}^{k} A^{(\theta_j)} = \begin{bmatrix} 1 & a_k(\theta_1, \ldots, \theta_k) \\ 0 & \prod_{j=1}^{k}(1-\theta_j^p) \end{bmatrix},$$

where

$$a_k(\theta_1, \ldots, \theta_k) = \theta_k + (1-\theta_k^p) a_{k-1}(\theta_1, \ldots, \theta_{k-1}), \tag{3.22}$$

being $a_{k-1}(\theta_1, \ldots, \theta_{k-1})$ the corresponding element in the factor $P_{k-1} = \prod_{j=0}^{k-1} A^{(\theta_j)} \in \Sigma_{k-1}(\mathcal{F})$.

If we consider the maximum norm in $\mathbb{C}^2$ defined by $\|x\|_\infty = \max\{|x_1|, |x_2|\}$, we obtain $\|P_k\|_\infty = 1 + a_k(\theta_1, \ldots, \theta_k)$. Therefore, with

$$\gamma_k = \max_{0\leq\theta_1,\ldots,\theta_k\leq 1} a_k(\theta_1, \ldots, \theta_k), \tag{3.23}$$

we can conclude that

$$\hat{\mu}_k(\mathcal{F}) = 1 + \gamma_k. \tag{3.24}$$

Now consider the two-variable function $\phi(\theta, \gamma) = \theta + (1-\theta^p)\gamma$, which is non decreasing with respect to γ for all $\theta \in [0, 1]$. If $\gamma_{k-1} \geq 1$, it is straightforward to see that $\max_{0\leq\theta\leq 1, 0\leq\gamma\leq\gamma_{k-1}} \phi(\theta, \gamma) = \phi(\theta_k^*, \gamma_{k-1})$, where $\theta_k^* = (p\gamma_{k-1})^{-1/(p-1)} < 1$. Thus, being $\gamma_1 = 1$, the recurrence relation (3.22) implies that the numbers (3.23) satisfy the recursion

$$\gamma_k = \gamma_{k-1} + c_p \gamma_{k-1}^{-1/(p-1)}, \quad \text{for } k \geq 2,$$

where $c_p = \frac{p-1}{p} p^{-1/(p-1)} < 1$. Finally, some standard computations allow us to state that the sequence $\{\gamma_k\}_{k\geq 0} \in \Theta(k^{(p-1)/p})$ and hence, by (3.24), we can conclude that

$$\text{ord}(\mathcal{F}) = \Theta(k^{(p-1)/p}).$$

4 On the Finiteness Conjecture for defective families

Let us consider the Finiteness Conjecture (Conjecture 1.1). It can be reformulated by saying that, if $\mathcal{F}$ is a finite family of complex $n \times n$-matrices, there exist $k^* \geq 1$ and a product $\bar{P} \in \Sigma_{k^*}(\mathcal{F})$ such that

$$\rho(\mathcal{F})^{1-k^*}\rho(\bar{P}) = \rho(\mathcal{F}), \tag{4.1}$$

which is equivalent to (1.6).

Definition 4.1 *If $\mathcal{F}$ is a bounded family of complex $n \times n$-matrices, any matrix $\bar{P} \in \Sigma_{k^*}(\mathcal{F})$ satisfying* (4.1) *for some $k^* \geq 1$ will be called a* spectrum-maximizing product *(in short, an* s.m.p.*) for $\mathcal{F}$.*

It is worth remarking that, assuming the validity of the Finiteness Conjecture, we can reformulate Proposition 3.1 as follows.

Proposition 4.1 *For any bounded family $\mathcal{F}$ of complex $n \times n$-matrices it holds that*

$$\mathrm{ord}(\mathcal{F}) \subseteq \Omega(k^{d^*}\rho(\mathcal{F})^k),$$

where $d^* = \max\{d_{\bar{P}} \mid \bar{P} \text{ s.m.p. for } \mathcal{F}\}$.

Now observe that, if $\mathcal{F}$ is nondefective, (4.1) is equivalent to

$$\mathrm{ord}(\rho(\mathcal{F})^{1-k^*}\bar{P}) = \mathrm{ord}(\mathcal{F}). \tag{4.2}$$

This easily follows from Corollary 3.1 since, by Proposition 4.1, the matrix $\bar{P}$ must be nondefective (otherwise the family $\mathcal{F}$ would be defective).

Definition 4.2 *If $\mathcal{F}$ is a bounded family of complex $n \times n$-matrices, any matrix $\hat{P} \in \Sigma_{k^*}(\mathcal{F})$ satisfying* (4.2) *for some $k^* \geq 1$ will be called an* order-maximizing product *(in short, an* o.m.p.*) for $\mathcal{F}$.*

Whereas the two concepts of s.m.p. and o.m.p. are equivalent for nondefective families, this is not the case for defective families. More precisely, we can say that, in general, given a nondefective family $\mathcal{F}$, an o.m.p. $\hat{P}$ is also an s.m.p., but not vice versa.

Now observe that the Finiteness Conjecture assumes the existence of at least one s.m.p. $\bar{P}$ for finite families, but does not assume the existence of an o.m.p. $\hat{P}$. In such a context, let us consider the following two examples.

Example 4.1 Consider the family $\mathcal{F} = \{A, B\}$ of the real 2×2-matrices

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}.$$

This family is normalized (i.e., $\rho(\mathcal{F}) = 1$). The matrices A and B are nondefective, but the product

$$\hat{P} = AB = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \in \Sigma_2(\mathcal{F})$$

is an s.m.p. and is defective. Therefore, by Proposition 4.1 and Theorem 3.1, it follows that $\mathrm{ord}(\hat{P}) = \mathrm{ord}(\mathcal{F}) = \Theta(k)$, i.e. $\hat{P}$ is an o.m.p. for $\mathcal{F}$.

Example 4.2 Consider the family $\mathcal{F} = \{A, B\}$ of the complex 2×2-matrices

$$A = \begin{bmatrix} 1 & 0 \\ 0 & e^i \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix},$$

where i is the imaginary unit. This family is normalized (i.e., $\rho(\mathcal{F}) = 1$). The matrices A and B are nondefective and, since $B^2 = I$ and $e^{ik} \neq -1$ for all $k \geq 1$, no product $P \in \Sigma(\mathcal{F})$ is defective either.

Nevertheless, it is wellknown that there exists an increasing sequence of integers k_s such that $\lim_{s\to\infty} e^{ik_s} = 1$ (see also the forecoming Lemma 5.2). Thus, since

$$P_s = A^{k_s-1}BAB = \begin{bmatrix} 1 & 1-e^i \\ 0 & e^{ik_s} \end{bmatrix},$$

there exists

$$\tilde{P} = \lim_{s\to\infty} P_s = \begin{bmatrix} 1 & 1-e^i \\ 0 & 1 \end{bmatrix},$$

which is a defective matrix.

Since the existence of such a defective limit matrix $\tilde{P}$ implies the unboundedness of $\Sigma(\mathcal{F})$ (see the forecoming Theorem 5.1 for the proof), we can conclude that no o.m.p. $\hat{P}$ exists for $\mathcal{F}$.

Example 4.3 Consider the family $\mathcal{F} = \{A, B\}$ of the real 3×3-matrices

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(1) & -\sin(1) \\ 0 & \sin(1) & \cos(1) \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Again, this family is normalized (i.e., $\rho(\mathcal{F}) = 1$). The matrices A and B are nondefective and, since $B^2 = I$ and the eigenvalues of the diagonal block $\begin{bmatrix} \cos(1) & -\sin(1) \\ \sin(1) & \cos(1) \end{bmatrix}$ in A are $\{e^i, e^{-i}\}$, no product $P \in \Sigma(\mathcal{F})$ is defective as in Example 4.2. Nevertheless, by using the same sequence of integers k_s, we obtain

$$P_s = A^{k_s - 1} BAB = \begin{bmatrix} 1 & 1-\cos(1) & \sin(1) \\ 0 & \cos(k_s) & -\sin(k_s) \\ 0 & \sin(k_s) & \cos(k_s) \end{bmatrix}$$

and, therefore, the defective limit matrix

$$\tilde{P} = \lim_{s \to \infty} P_s = \begin{bmatrix} 1 & 1-\cos(1) & \sin(1) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Again, the existence of such a defective limit matrix $\tilde{P}$ implies the unboundedness of $\Sigma(\mathcal{F})$ and, thus, we can conclude that no o.m.p. $\hat{P}$ exists for $\mathcal{F}$.

Remark 4.1 *Examples* 4.2 *and* 4.3 *guarantee that a stronger reformulation of the Finiteness Conjecture which always assumes the existence of at least an o.m.p.* $\hat{P}$ *for finite families is false.*

5 Defectivity and limit maximizing products

In this section, we shall prove that, for finite defective families of 2×2-matrices, the essence of all the possible cases is illustrated by Examples 4.1 and 4.2.

Definition 5.1 *Assume that* $\mathcal{F}$ *is a normalized bounded family of complex* $n \times n$*-matrices (i.e.,* $\rho(\mathcal{F}) = 1$*) and that there exists a sequence of products* $P_s \in \Sigma_{k_s}(\mathcal{F})$*,* k_s *non decreasing integers, such that*

$$\lim_{s \to \infty} P_s = \tilde{P}, \tag{5.1}$$

where $\rho(\tilde{P}) = 1$. Then $\tilde{P}$ will be called a limit spectrum-maximizing product *(in short, an* l.s.m.p.*) for the normalized family $\mathcal{F}$.*

Note that, for a normalized family, an s.m.p. $\bar{P}$ is an l.s.m.p., too. To see this, just put $P_s = \bar{P}$ for all $s \geq 1$.

Theorem 5.1 *Let $\mathcal{F}$ be a bounded family of complex $n \times n$-matrices. If there exists a defective l.s.m.p. $\tilde{P}$ for the normalized family $\mathcal{F}'$, then $\mathcal{F}$ is defective.*

Proof. Since $\tilde{P}$ is defective, for any constant $C > 0$ and any induced matrix norm $\|\cdot\|$, there exists an integer $r_C \geq 1$ such that

$$\|\tilde{P}^{r_C}\| \geq C.$$

On the other hand, (5.1) implies that $\lim_{s\to\infty} P_s^{r_C} = \tilde{P}^{r_C}$. Therefore, there exists an integer $s_C \geq 1$ such that

$$\|P_{s_C}^{r_C} - \tilde{P}^{r_C}\| \leq C/2.$$

Thus we can conclude that

$$\|P_{s_C}^{r_C}\| \geq C/2,$$

that is $\Sigma(\mathcal{F}')$ is unbounded. ∎

In the light of Example 3.1, it is evident that the converse of Theorem 5.1 does not hold for infinite families of matrices. In fact, it is immediate to realize that no l.s.m.p. $\tilde{P}$ exists for the normalized family $\mathcal{F}$ considered there.

The rest of this section is devoted to prove the converse of Theorem 5.1 in the case of finite families of 2×2-matrices. For finite families of $n \times n$-matrices with $n \geq 3$ the problem is much more difficult to analyze and, so far, we have not been able to solve it.

Lemma 5.1 *Let $\mathcal{G} = \left\{\{A^{(i)}\}_{i\in\mathcal{I}}, \{B^{(j)}\}_{j\in\mathcal{J}}\right\}$ be a normalized bounded family of complex 2×2-matrices (i.e., $\rho(\mathcal{G}) = 1$), where*

$$A^{(i)} = \begin{bmatrix} \tilde{u}_i & \tilde{\alpha}_i \\ 0 & \tilde{a}_i \end{bmatrix} \quad \text{and} \quad B^{(j)} = \begin{bmatrix} \tilde{b}_j & \tilde{\beta}_j \\ 0 & \tilde{v}_j \end{bmatrix}.$$

If there exist two nonnegative real numbers $a < 1$ and $b < 1$ such that $|\tilde{a}_i| \leq a$ for all $i \in \mathcal{I}$ and $|\tilde{b}_j| \leq b$ for all $j \in \mathcal{J}$, then the family $\mathcal{G}$ is nondefective.

Proof. Since $\mathcal{G}$ is normalized and the matrices are all upper triangular, it holds that $|\tilde{u}_i| \le 1$ for all $i \in \mathcal{I}$ and $|\tilde{v}_j| \le 1$ for all $j \in \mathcal{J}$.

Then consider the family $\hat{\mathcal{G}} = \left\{ \{\hat{A}^{(i)}\}_{i\in\mathcal{I}}, \{\hat{B}^{(j)}\}_{j\in\mathcal{J}} \right\}$ of real 2×2-matrices with nonnegative elements associated to $\mathcal{G}$, where

$$\hat{A}^{(i)} = \begin{bmatrix} 1 & \alpha_i \\ 0 & a_i \end{bmatrix} \quad \text{and} \quad \hat{B}^{(j)} = \begin{bmatrix} b_j & \beta_j \\ 0 & 1 \end{bmatrix},$$

with $\alpha_i = |\tilde{\alpha}_i|$, $a_i = |\tilde{a}_i|$, $\beta_j = |\tilde{\beta}_j|$ and $b_j = |\tilde{b}_j|$.

Clearly, the family $\hat{\mathcal{G}}$ is normalized too, and, for any $k \ge 1$ and product $P \in \Sigma_k(\mathcal{G})$, the corresponding product $\hat{P} \in \Sigma_k(\hat{\mathcal{G}})$ is such that $|P| \le \hat{P}$, where the inequality has to be understood elementwise and $|P|$ is the matrix whose elements are the moduli of the corresponding elements in P. Therefore, if $\Sigma(\hat{\mathcal{G}})$ is bounded (i.e., if $\hat{\mathcal{G}}$ is nondefective), then $\Sigma(\mathcal{G})$ is bounded (i.e., $\mathcal{G}$ is nondefective), too.

In the light of Proposition 2.2, we shall prove the boundedness of $\Sigma(\hat{\mathcal{G}})$ by constructing an extremal norm $\|\cdot\|_*$ for the family $\hat{\mathcal{G}}$.

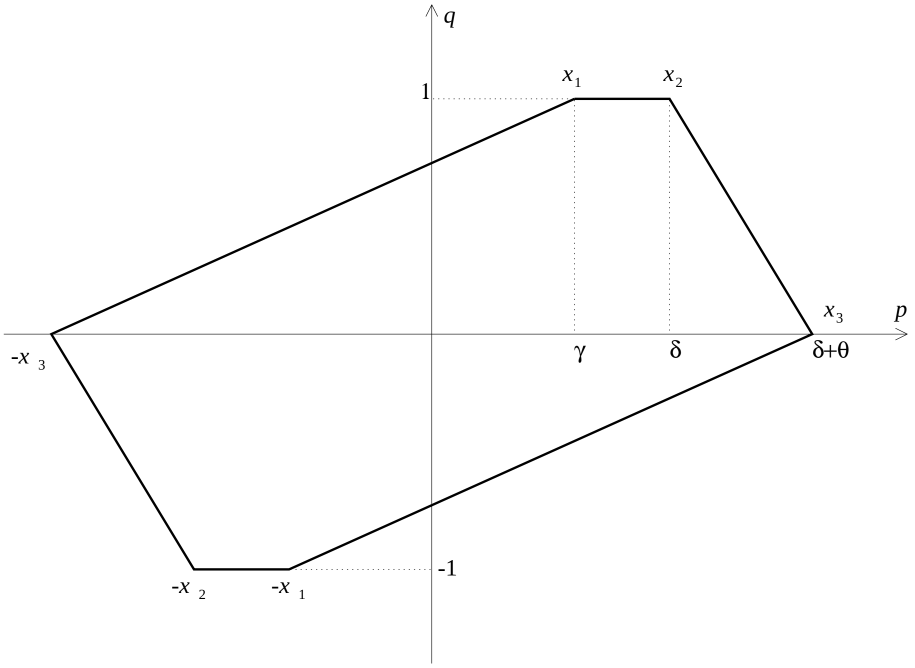

Fig. 1: Unit ball $\mathcal{P}$ of the norm $\|\cdot\|_*$.

To this aim we set

$$\theta = \sup_{i \in \mathcal{I}} \frac{\alpha_i}{1-a_i}, \quad \gamma = \inf_{j \in \mathcal{J}} \frac{\beta_j}{1-b_j}, \quad \delta = \sup_{j \in \mathcal{J}} \frac{\beta_j}{1-b_j},$$

and define the vectors

$$x_1 = \begin{bmatrix} \gamma \\ 1 \end{bmatrix}, \quad x_2 = \begin{bmatrix} \delta \\ 1 \end{bmatrix}, \quad x_3 = \begin{bmatrix} \delta + \theta \\ 0 \end{bmatrix}.$$

Then we consider the polytope $\mathcal{P}$ whose vertices are the vectors x_1, x_2, x_3, $-x_1$, $-x_2$, $-x_3$ as shown in Fig. 1, and denote by $\|\cdot\|_*$ the norm whose unit ball is $\mathcal{P}$.

In order to prove that $\|\cdot\|_*$ is an extremal norm for the family $\hat{\mathcal{G}}$, it is sufficient to verify that the vertices of the polytope $\mathcal{P}$ are mapped into $\mathcal{P}$ itself by all of the matrices of $\hat{\mathcal{G}}$.

Indeed, we have

$$\hat{A}^{(i)} x_1 = \begin{bmatrix} \gamma + \alpha_i \\ a_i \end{bmatrix}, \quad \hat{A}^{(i)} x_2 = \begin{bmatrix} \delta + \alpha_i \\ a_i \end{bmatrix}, \quad \hat{A}^{(i)} x_3 = x_3,$$

$$\hat{B}^{(j)} x_1 = \begin{bmatrix} b_j \gamma + \beta_j \\ 1 \end{bmatrix}, \quad \hat{B}^{(j)} x_2 = \begin{bmatrix} b_j \delta + \beta_j \\ 1 \end{bmatrix}, \quad \hat{B}^{(j)} x_3 = b_j x_3,$$

and simple but tedious calculations show that these vectors belong to $\mathcal{P}$ for all $i \in \mathcal{I}$ and for all $j \in \mathcal{J}$. ■

Lemma 5.2 *Let $\mathcal{H} = \{A, B\}$ be a normalized family of complex 2×2-matrices such that, for some nonsingular complex 2×2-matrix L,*

$$L^{-1} A L = u_A \begin{bmatrix} 1 & 0 \\ 0 & a_{22} \end{bmatrix} \quad \text{and} \quad L^{-1} B L = u_B \begin{bmatrix} 1 & b_{12} \\ 0 & b_{22} \end{bmatrix},$$

where $|u_A| = |u_B| = |a_{22}| = |b_{22}| = 1$, $a_{22} \neq 1$ and $b_{12} \neq 0$. Then there exists a defective l.s.m.p. $\tilde{P}$ for $\mathcal{H}$.

Proof. If $b_{22} = 1$, then the matrix B itself is a defective s.m.p. and the result is trivial. Therefore, assume that $b_{22} \neq 1$.

If there exist two integers $k, h \geq 1$ such that

$$a_{22}^k b_{22}^h = 1, \tag{5.2}$$

then two cases are possible: either both $a_{22}^k \neq 1$ and $b_{22}^h \neq 1$ or $a_{22}^k = b_{22}^h = 1$. In the former case, since $b_{12} \neq 0$, the matrix

$$\bar{P} = A^k B^h = u_A^k u_B^h L \begin{bmatrix} 1 & b_{12}\frac{1-b_{22}^h}{1-b_{22}} \\ 0 & 1 \end{bmatrix} L^{-1}$$

is a defective s.m.p. In the latter case, since $a_{22} \neq 1$, it is the matrix

$$\bar{P} = A^{k-1} B^{h-1} AB = u_A^k u_B^h L \begin{bmatrix} 1 & b_{12}(1 - a_{22}) \\ 0 & 1 \end{bmatrix} L^{-1}$$

which is a defective s.m.p. However, in both cases, the result is proved.

Then we are left to consider

$$a_{22}^k b_{22}^h \neq 1 \quad \text{for all} \quad k, h \geq 1, \tag{5.3}$$

which implies $a_{22}^k \neq 1$ for all $k \geq 1$ and/or $b_{22}^h \neq 1$ for all $h \geq 1$. In any case, there exist increasing sequences of integers k_s and h_s such that

$$\lim_{s\to\infty} a_{22}^{k_s} = 1 \quad \text{and} \quad \lim_{s\to\infty} b_{22}^{h_s} = 1. \tag{5.4}$$

More precisely, Theorem 193 in Hardy and Wright [HW79] immediately implies that the integers k_s and h_s may be assumed to satisfy the inequalities

$$|a_{22}^{k_s} - 1| \leq \frac{2\pi}{\sqrt{5}k_s} \quad \text{and} \quad |b_{22}^{h_s} - 1| \leq \frac{2\pi}{\sqrt{5}h_s}. \tag{5.5}$$

Therefore, by (5.4), the products

$$P_s = A^{k_s-1} B^{h_s-1} AB = u_A^{k_s} u_B^{h_s} L \begin{bmatrix} 1 & b_{12}\left(1 - a_{22}\frac{b_{22}^{h_s} - b_{22}}{1-b_{22}}\right) \\ 0 & a_{22}^{k_s} b_{22}^{h_s} \end{bmatrix} L^{-1} \tag{5.6}$$

are such that there exists

$$P^* = \lim_{s\to\infty} u_A^{-k_s} u_B^{-h_s} P_s = L \begin{bmatrix} 1 & b_{12}(1 - a_{22}) \\ 0 & 1 \end{bmatrix} L^{-1},$$

which is a defective matrix.

Moreover, since the sequence of numbers $\{u_A^{k_s} u_B^{h_s}\}_{s\geq 1}$ lies on the unit circle of the complex plane, there exists a subsequence $\{u_A^{k_{s_r}} u_B^{h_{s_r}}\}_{r\geq 1}$ such that $\lim_{r\to\infty} u_A^{k_{s_r}} u_B^{h_{s_r}} = u$ with $|u| = 1$. Therefore, we can conclude that

$$\lim_{r\to\infty} P_{s_r} = \tilde{P} = uP^*,$$

i.e. an l.s.m.p. for the family $\mathcal{H}$. ■

Remark 5.1 *If* (5.2) *holds, then there exists a defective s.m.p.* $\bar{P}$*. Therefore, Theorem* 3.1 *and Proposition* 4.1 *imply that* $\mathrm{ord}(\mathcal{H}) = \Theta(k)$ *and that* $\bar{P}$ *is an o.m.p. for* $\mathcal{H}$*.*

On the contrary, if (5.3) *holds,* $\mathrm{ord}(\mathcal{H})$ *is not easy to evaluate, even taking* (5.5) *into account.*

Now we are in a position to prove the main result of this section.

Theorem 5.2 *Let* $\mathcal{F}$ *be a finite defective family of complex* 2×2*-matrices. Then there exists a defective l.s.m.p.* $\tilde{P}$ *for the normalized family* $\mathcal{F}'$*.*

Proof. Since the family $\mathcal{F} = \{A^{(i)}\}_{i=1}^m$ is defective, Theorem 2.1 assures the existence of a nonsingular 2×2-matrix M such that

$$M^{-1}A^{(i)}M = u_i\rho(\mathcal{F}) \begin{bmatrix} a_{11}^{(i)} & a_{12}^{(i)} \\ 0 & a_{22}^{(i)} \end{bmatrix}, \quad i = 1,\ldots,m,$$

with $|u_i| = 1$ and

$$a_{11}^{(i)} \in \mathbb{R},\ 0 \leq a_{11}^{(i)} \leq 1 \quad \text{and} \quad |a_{22}^{(i)}| \leq 1, \quad i = 1,\ldots,m. \tag{5.7}$$

Moreover, we can assume that

$$M^{-1}A^{(i)}M \neq u_i\rho(\mathcal{F})I, \quad i = 1,\ldots,m, \tag{5.8}$$

where I is the identity matrix. Otherwise, to our aims, we could equivalently consider the subfamily of $\mathcal{F}$ which is obtained just by suppressing the matrices of the form $u_i\rho(\mathcal{F})I$.

In view of (5.7), we have that at least one matrix of $\mathcal{F}$, say $A^{(1)}$, satisfies the condition

$$a_{11}^{(1)} = 1 \quad \text{and} \quad |a_{22}^{(1)}| = 1. \tag{5.9}$$

Otherwise, $\mathcal{F}$ being finite, the hypotheses of Lemma 5.1 would hold for the normalized family $\mathcal{F}'$, yielding the nondefectivity of $\mathcal{F}$, that makes the absurde.

If $A^{(1)}$ is defective, the result is trivial since, in this case, $A^{(1)}$ itself is a defective s.m.p. for $\mathcal{F}$. Thus we assume that $A^{(1)}$ is nondefective and, therefore, in the light of (5.8), that

$$a_{22}^{(1)} \neq 1. \tag{5.10}$$

So $M^{-1}A^{(1)}M$ may be reduced to the diagonal form

$$\bar{A}^{(1)} = U^{-1}M^{-1}A^{(1)}MU = u_1\rho(\mathcal{F})\begin{bmatrix} 1 & 0 \\ 0 & a_{22}^{(1)} \end{bmatrix}, \tag{5.11}$$

where

$$U = \begin{bmatrix} 1 & -\frac{a_{12}^{(1)}}{1-a_{22}^{(1)}} \\ 0 & 1 \end{bmatrix}.$$

Now consider the family $\bar{\mathcal{F}}$, obtained by applying the similarity transformation determined by $L = MU$ to all the matrices of $\mathcal{F}$. Thus we obtain the similar matrices

$$\bar{A}^{(i)} = L^{-1}A^{(i)}L = u_i\rho(\mathcal{F})\begin{bmatrix} a_{11}^{(i)} & \bar{a}_{12}^{(i)} \\ 0 & a_{22}^{(i)} \end{bmatrix}, \quad i = 1, \dots, m, \tag{5.12}$$

where

$$\bar{a}_{12}^{(i)} = \frac{a_{12}^{(i)}(1 - a_{22}^{(1)}) - a_{12}^{(1)}(a_{11}^{(i)} - a_{22}^{(i)})}{1 - a_{22}^{(1)}}.$$

As in the proof of Lemma 5.1, we then consider the matrices $|\bar{A}^{(i)}|$, whose elements are the moduli of the corresponding elements in $\bar{A}^{(i)}$.

Since $\Sigma(\mathcal{F}')$ is unbounded, so is $\Sigma(\bar{\mathcal{F}}')$ and, a fortiori, $\Sigma(\mathcal{G}')$, where

$$\mathcal{G} = \{|\bar{A}^{(i)}| \mid \bar{A}^{(i)} \in \bar{\mathcal{F}} \text{ and } |\bar{A}^{(i)}| \neq \rho(\mathcal{F})I\}.$$

By (5.9) and (5.11), we have that $|\bar{A}^{(1)}| = \rho(\mathcal{F})I$ and, hence, that $|\bar{A}^{(1)}| \notin \mathcal{G}$.

Using again Lemma 5.1 for the finite normalized family $\mathcal{G}'$ yields the existence of a matrix, say $|\bar{A}^{(2)}|$, such that

$$a_{11}^{(2)} = 1 \quad \text{and} \quad |a_{22}^{(2)}| = 1. \tag{5.13}$$

Moreover, being $|\bar{A}^{(2)}| \neq \rho(\mathcal{F})I$, it holds that

$$\bar{a}_{12}^{(2)} \neq 0, \tag{5.14}$$

too.

By (5.11), (5.9), (5.10), (5.12), (5.13) and (5.14), we can conclude that the subfamily $\mathcal{H} = \{\rho(\mathcal{F})^{-1}A^{(1)}, \rho(\mathcal{F})^{-1}A^{(2)}\}$ of the normalized family $\mathcal{F}'$ satisfies the hypotheses of Lemma 5.2. Therefore, there exists an l.s.m.p. $\tilde{P}$ for $\mathcal{F}'$. ■

Remark that, if the family $\mathcal{F}$ were infinite, the proof would not work. In fact, we could not prove formulae (5.9) and (5.13).

We conclude this section by particularizing the results of Lemma 5.2 and Theorem 5.2 to the special case of real 2×2-matrices.

Lemma 5.3 *Let $\mathcal{H} = \{A, B\}$ be a normalized family of real 2×2-matrices such that, for some nonsingular complex 2×2-matrix L,*

$$L^{-1}AL = u_A \begin{bmatrix} 1 & 0 \\ 0 & a_{22} \end{bmatrix} \quad \text{and} \quad L^{-1}BL = u_B \begin{bmatrix} 1 & b_{12} \\ 0 & b_{22} \end{bmatrix}, \tag{5.15}$$

where $|u_A| = |u_B| = |a_{22}| = |b_{22}| = 1$, $a_{22} \neq 1$ and $b_{12} \neq 0$. Then

$$A = \pm L \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} L^{-1}. \tag{5.16}$$

Moreover, either

$$B = \pm L \begin{bmatrix} 1 & b_{12} \\ 0 & 1 \end{bmatrix} L^{-1} \tag{5.17}$$

is defective or

$$B = \pm L \begin{bmatrix} 1 & b_{12} \\ 0 & -1 \end{bmatrix} L^{-1} \tag{5.18}$$

and AB is defective.

In both cases there exists an o.m.p. $\hat{P}$ for $\mathcal{H}$ (either $\hat{P} = B$ or $\hat{P} = AB$) and $\text{ord}(\mathcal{H}) = \Theta(k)$.

Proof. Consider the matrix A. Since it is real with eigenvalues of unitary modulus different from each other, two only cases are possible: the eigenvalues are either $\{1, -1\}$ or $\{e^{i\theta}, e^{-i\theta}\}$ with $\theta \in (0, 2\pi), \theta \neq \pi$. However, in both cases, the columns of the matrix L in (5.15) are necessarily the eigenvectors of A.

Assume first that the eigenvalues of A are $\{1, -1\}$. Then we necessarily have that $L = \begin{bmatrix} \alpha p_{11} & \beta p_{12} \\ \alpha p_{21} & \beta p_{22} \end{bmatrix}$, where $p_{ij} \in \mathbb{R}$ and $\alpha, \beta \in \mathbb{C}$. Moreover, (5.16) must hold, too.

Since B is real, it is immediately seen that both the diagonal elements of the upper triangular matrix $L^{-1}BL$ are real, i.e. 1 or -1. This means that either (5.17) or (5.18) must hold. In the former case B itself is a defective s.m.p., whereas, in the latter case, $AB = \pm L \begin{bmatrix} 1 & b_{12} \\ 0 & 1 \end{bmatrix} L^{-1}$ is a defective s.m.p.

In both cases Theorem 3.1 and Proposition 4.1 imply that $\mathrm{ord}(\mathcal{H}) = \Theta(k)$ and that either $\hat{P} = B$ or $\hat{P} = AB$ is an o.m.p. for $\mathcal{H}$.

In order to conclude the proof, we are going to show that the eigenvalues of A can not be $\{e^{i\theta}, e^{-i\theta}\}$ with $\theta \in (0, 2\pi), \theta \neq \pi$. In fact, if it were so, it should be $L = \begin{bmatrix} \alpha(p_1 + iq_1) & \beta(p_1 - iq_1) \\ \alpha(p_2 + iq_2) & \beta(p_2 - iq_2) \end{bmatrix}$, where $p_i, q_i \in \mathbb{R}$ and $\alpha, \beta \in \mathbb{C}$. Since B is real, it is immediately seen that the two extra-diagonal elements of the matrix $L^{-1}BL$ would be complex conjugate. But this makes the absurde, since $L^{-1}BL$ is upper triangular with $b_{12} \neq 0$. ■

Theorem 5.3 *Let $\mathcal{F}$ be a finite defective family of real 2×2-matrices. Then there exists an o.m.p. $\hat{P}$ for $\mathcal{F}$ and* $\mathrm{ord}(\mathcal{F}) = \Theta(k\rho(\mathcal{F})^k)$.

Proof. In order to prove that an o.m.p. $\hat{P}$ exists for $\mathcal{F}$, we can proceed exactly in the same way as for the previous Theorem 5.2 but in the last step, where Lemma 5.3 applies in place of Lemma 5.2. Finally, the link between $\mathcal{F}$ and its normalized family $\mathcal{F}'$ yields the equality $\mathrm{ord}(\mathcal{F}) = \Theta(k\rho(\mathcal{F})^k)$. ■

Remark that, in the light of Example 4.3, the foregoing result does not extend to families of real $n \times n$-matrices with $n \geq 3$.

6 Conclusions

In this paper we have drawn our attention to the case of *defective* families of matrices (see Definition 2.3).

By using the concept of *asymptotic order* (see Definition 3.1), we have tried to get more information about the possible asymptotic behaviour of the norm of the products of defective families as the number of factors goes to infinity. In particular, we have found some general lower and upper bounds to such behaviour (see Propositions 3.1, 3.2, 3.3, and Theorem 3.1).

We have introduced the concepts of *spectrum-maximizing product* and *order-maximizing product* and observed that the Finiteness Conjecture (Conjecture 1.1) assumes the existence of spectrum-maximizing products for any finite family of matrices. Then, in the light of Examples 4.2 and 4.3, we have concluded that a possible stronger reformulation of the Finiteness Conjecture, which assumes also the existence of order-maximizing products, is false.

Finally, we have also introduced the concept of *limit spectrum-maximizing product* for normalized families of matrices and we have seen that the existence of defective such limit products implies defectivity (see Theorem 5.1). The converse implication, which, in any case, might hold only for finite families (Example 3.1 is a suitable counterexample), is much more difficult to prove. We have been able to do it only for finite families of 2×2-matrices (see Theorem 5.2). Although our intuition suggests that, at least to some extent, the result should generalize to all finite families of $n \times n$-matrices, we have not been able to fix the problem yet.

The conclusion is that the asymptotic order of a defective family of matrices is, in the general case, difficult to compute. In particular, Example 3.1 shows that it may well be different from the asymptotic order of a single matrix, which, on the contrary, is always of the type displayed in formula (3.5).

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