Numerical stability of nonlinear delay differential equations of neutral type

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Abstract

This paper is devoted to the stability analysis of both the true solutions and the numerical approximations for nonlinear systems of neutral delay differential equations (NDDEs) of the form $y'(t) = F\left(t, y(t), G\left(t, y(t-\tau(t)), y'(t-\tau(t))\right)\right)$.

This work extends the results recently obtained by the authors [2] for the linear case. This is accomplished by considering a suitable reformulation of the given system, which transforms it into a nonlinear differential system coupled with an algebraic functional recursion. Numerical processes preserving the qualitative properties of the solutions are also investigated.

Key words: Neutral delay differential equations, numerical stability, RK methods for delay equations.

1 Introduction

Delay differential equations (DDE) are assuming an increasingly important role in many applied disciplines. One of the reasons is that progress has been made in the mathematical understanding and theory of DDEs. Further, a multitude of different interesting problems lead to DDEs in different fields like biology, economy, circuit theory, control theory and electrodynamics (see, e.g., [4,10,14,7]). The mathematical theory necessary for the efficient solution and understanding of key issues like convergence and stability has been advanced especially in the last few years. A comprehensive introduction to the subject of DDEs and numerical solvers is given in a review paper by Baker, Paul and Willé [3] and in a book chapter by Zennaro [19].

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Our work was in part motivated by the solution of circuit theory time domain problems which lead to DDEs. Two important examples are the method of characteristics transmission line models [4] and the partial element equivalent circuit (PEEC) three dimensional electromagnetic circuit model [14]. Recently, implementations of the PEEC approach have shown to be promising for the time domain analysis of electromagnetic interactions of packaged electronics [13]. The key feature of these problems is the fact that they involve delayed sources which are of the form $y(t-\tau)$ and $y'(t-\tau)$.

In this paper we investigate neutral delay differential equations (NDDEs) of the form

$$\begin{cases} y'(t) = F\Big(t, y(t), G\big(t, y(t-\tau(t)), y'(t-\tau(t))\big)\Big), & t \geq t_0, \\ y(t) = g(t), & t \leq t_0, \end{cases} \tag{1}$$

where F and G are complex continuous vector functions, $g(t)$ is a C^1-continuous complex-valued function and $\tau(t)$ is a continuous delay function such that

(H1) $\quad \tau(t) \geq \tau_0 > 0$ and $\alpha(t) = t - \tau(t)$ is increasing $\forall t \geq t_0$.

We first examine sufficient conditions for the contractivity and for the asymptotic stability of the true solutions of (1), which represents the model for the system to be studied. From a computational point of view, we are also interested in the qualitative behaviour of the numerical solution of the NDDE. For this, in the second part of the paper, we investigate both the contractivity and the asymptotic stability of the numerical solution furnished by RK methods.

Confined to the linear case, several researchers have studied the asymptotic behaviour of the true solutions (see, e.g., Li [12]) as well as the numerical solutions (see, e.g., Brayton & Willoughby [5], Hu & Mitsui [8], Kuang et al.[11] and, quite recently, Bellen et al.[2]). Relevant to the considered nonlinear test problem (1), instead, only very few results dealing with the qualitative behaviour of both the true and the numerical solutions have been published. Among them, we address the reader to the recent paper by Torelli and Vermiglio [17], where - however - both the model and the used approach are differerent with respect to those proposed here.

Our approach is based on the contractivity properties of the solutions of (1) and we extend the contractivity requirements to the numerical solutions. For the case of (non neutral) nonlinear delay differential equations, this kind of methodology has been first introduced by Torelli [15,16] and then developed by Bellen & Zennaro [6], Bellen [1] and Zennaro [18,20].

2 The standard approach

The most standard approach considered in the literature consists in integrating (1) step-by-step. To this aim we need to determine the breaking points of the solution $y(t)$, which are points associated with discontinuities in the derivatives of $y(t)$, due to the presence of the functional argument $\alpha(t)$. By hypothesis (H1) on the delay function, we label these points as

$$\xi_0 = t_0 < \xi_1 < \cdots < \xi_n < \xi_{n+1} < \cdots,$$

ξ_{n+1} being the unique solution of $\alpha(\xi) = \xi_n$. Then we define the intervals $I_0 := [-\tau(t_0), \xi_0]$ and $I_n = [\xi_{n-1}, \xi_n]$, for $n \geq 1$. The analysis of the behaviour of the solutions can be done across the intervals I_n, by relating the solution in I_n with the one in I_{n-1}.

The classical way of numerically solving the system (1) consists in making use of numerical methods for ODEs to integrate the system

$$y'(t) = F\Big(t, y(t), G\big(t, \eta(\alpha(t)), \zeta(\alpha(t))\big)\Big), \qquad t \in I_n,$$

where $\eta(s)$ and $\zeta(s)$ are approximations of $y(s)$ and $y'(s)$ respectively, obtained by the numerical method itself in the previous interval I_{n-1}.

3 A suitable reformulation of the problem

In this work we propose a suitable reformulation of the problem which, apparently, does not require to approximate the derivative of the numerical solution. In order to do this, we rewrite the problem (1) as follows:

$$\begin{cases} y'(t) = F\left(t, y(t), \Phi(t)\right) & t \geq t_0, \\ y(t) = g(t), & t \leq t_0, \end{cases} \tag{2}$$

where

$$\Phi(t) = \begin{cases} G\big(t, g(\alpha(t)), g'(\alpha(t))\big), & \text{if } t_0 \leq t < \xi_1, \\ G\Big(t, y(\alpha(t)), F\big(\alpha(t), y(\alpha(t)), \Phi(\alpha(t))\big)\Big) & \text{if } t \geq \xi_1. \end{cases} \tag{3}$$

Observe that $\Phi(t)$ is not continuous from the left at the breaking points ξ_k. In this approach the neutral system is transformed into an ordinary differential system plus an algebraic functional recursion. Consequently new numerical schemes for the approximation of the solution are suggested.

Without loss of generality, we assume that

$$F\,(t,0,0) \equiv G\,(t,0,0) \equiv 0,$$

so that the system

$$\begin{cases} y'(t) = F\Big(t, y(t), G\big(t, y(\alpha(t)), y'(\alpha(t))\big)\Big), & t \geq t_0, \\ y(t) \;= 0, & t \leq t_0, \end{cases}$$

has the trivial solution $y(t) \equiv 0$. Our stability analysis is based on the contractivity properties of the solutions of the ordinary differential equation (ODE)

$$\begin{cases} y'(t) \;= F\,(t, y(t), \Phi(t)) & t \geq t_0, \\ y(t_0) = y_0, \end{cases} \tag{4}$$

with respect to the forcing term $\Phi(t)$. Preliminarily, we make some standard assumptions. Given an inner product $< \cdot, \cdot >$ in $\mathbb{C}^m$ and the corresponding norm $\| \cdot \|$, we assume that $F : [t_0, +\infty) \times \mathbb{C}^m \times \mathbb{C}^m \to \mathbb{C}^m$ is continuous with respect to t and uniformly Lipschitz continuous with respect to the second and third variables, that is there exist two continuous functions $Y(t)$ and $W(t)$ such that

$$Y(t) \geq \sup_{x, y_1 \neq y_2} \frac{\Re\,[< F(t, y_1, x) - F(t, y_2, x), y_1 - y_2 >]}{\| \, y_1 - y_2 \, \|^2}, \tag{5}$$

$$W(t) \geq \sup_{y, x_1 \neq x_2} \frac{\| \, F(t, y, w_1) - F(t, y, w_2) \, \|}{\| \, w_1 - w_2 \, \|}. \tag{6}$$

In the sequel it will be convenient to consider the function

$$H\Big(t, x, z\Big) = G\Big(t, x, F\Big(\alpha(t), x, z\Big)\Big). \tag{7}$$

Similarly to what done for F, we assume for H a continuous dependence with respect to t and a uniform Lipschitz continuity with respect to x and z, that is there exist two continuous functions $X(t)$ and $Z(t)$ such that

$$X(t) \geq \sup_{z, x_1 \neq x_2} \frac{\| H(t, x_1, z) - H(t, x_2, z) \|}{\| x_1 - x_2 \|}, \tag{8}$$

$$Z(t) \geq \sup_{x, z_1 \neq z_2} \frac{\| H(t, x, z_1) - H(t, x, z_2) \|}{\| z_1 - z_2 \|}. \tag{9}$$

Then recall the following result (see, for example, Zennaro [20]).

Lemma 1 *Consider the system (4) where the components of the forcing term $\Phi(t)$ are assumed to be continuous functions. Moreover, chosen a suitable inner product $< \cdot, \cdot >$ such that (5) and (6) hold, assume that*

$$Y(t) \leq 0, \qquad t \geq t_0$$

and, for a bounded function $r(t) \geq 0$,

$$W(t) = -\, r(t)\, Y(t), \qquad t \geq t_0.$$

Then, for all $t \geq t_0$, the inequality

$$\| y(t) \| \leq E(t_0, t) \ \| y(t_0) \| + \left(1 - E(t_0, t)\right) \sup_{t_0 \leq s \leq t} (r(s) \ \| \Phi(s) \|) \tag{10}$$

holds, where

$$E(t_1, t_2) = \exp\left(\int_{t_1}^{t_2} Y(s)\, ds\right) \ \leq 1 \qquad \forall\, t_2 \geq t_1.$$

By using the foregoing result, it is not difficult to modify the proofs of Theorems 3.1 and 3.2 in [2] in order to extend the contractivity and asymptotic stability results from the linear to the nonlinear case (1) considered here. In the sequel we shall assume the function $\Phi(t)$ continuous in the closed interval $[\xi_{n-1}, \xi_n]$, $n \geq 1$, by considering

$$\Phi(\xi_n) = \lim_{t \to \xi_n^-} \Phi(t),$$

which exists. Therefore, there exists $\max\limits_{\xi_{n-1} \leq s \leq \xi_n} \| \Phi(s) \|$ for every $n \geq 1$. The following theorem concerns the contractivity properties of the solutions of (2)-(3). For technical reasons we shall assume $r(t) > 0$. In fact we shall consider the ratio $\frac{r(t)}{r(\alpha(t))}$.

Theorem 2 *Assume that the functions (5), (6), (8) and (9) fulfil the inequalities $Y(t) \le 0$, $r(t) > 0$ and*

$$r(t)\left(X(t) + \frac{Z(t)}{r(\alpha(t))}\right) \le 1, \qquad \forall t \ge t_0. \tag{11}$$

Then the solution $y(t)$ of (1) is such that

$$\| y(t) \| \le \max\left\{\| g(t_0) \|, \kappa\right\}, \tag{12}$$

where

$$\kappa = \sup_{t_0 \le s \le \xi_1} \| r(s)\, G\Big(s, g(\alpha(s)), g'(\alpha(s))\Big) \| \tag{13}$$

for every initial function $g(t)$ and for every delay $\tau(t)$ satisfying the assumption (H1).

PROOF. We proceed by developing a step-by-step analysis of the system (2)-(3) over the intervals $\{I_n\}$. Thus consider the interval $I_n = [\xi_{n-1}, \xi_n]$ $(n \ge 1)$. By Lemma 1, for every $t \in I_n$, we have

$$\| y(t) \| \le E(\xi_{n-1}, t)\ \| y(\xi_{n-1}) \| + (1 - E(\xi_{n-1}, t)) \sup_{\xi_{n-1} \le s \le t} (r(s)\ \| \Phi(s) \|), \tag{14}$$

where $E(\xi_{n-1}, t) = \exp\left(\int\limits_{\xi_{n-1}}^{t} Y(s)\, ds\right) \le 1$. Since the right-hand side of (14) is a convex combination, we immediately have

$$\| y(t) \| \le \max\left\{\| y(\xi_{n-1}) \|, \sup_{\xi_{n-1} \le s \le t} (r(s)\ \| \Phi(s) \|)\right\}. \tag{15}$$

Furthermore, by (3) and (7), it holds that

$$\| \Phi(t) \| = \begin{cases} \| G\Big(t, g(\alpha(t)), g'(\alpha(t))\Big) \|, & n = 1, \\ \| H\Big(t, y(\alpha(t)), \Phi(\alpha(t))\Big) \|, & n \ge 2. \end{cases} \tag{16}$$

Therefore, for $n \ge 2$, since $H(t, 0, 0) \equiv 0$, we have

$$
\begin{aligned}
\| \Phi(t) \| &= \| H\Big(t, y(\alpha(t)), \Phi(\alpha(t))\Big) \| - \| H\,(t,0,0) \| \\
&= \| H\Big(t, y(\alpha(t)), \Phi(\alpha(t))\Big) \| - \| H\Big(t, 0, \Phi(\alpha(t))\Big) \| + \\
&\quad \| H\Big(t, 0, \Phi(\alpha(t))\Big) \| - \| H\,(t,0,0) \| \\
&\leq X(t) \ \| y(\alpha(t)) \| + Z(t) \ \| \Phi(\alpha(t)) \|
\end{aligned}
\tag{17}
$$

and, hence, assumption (11) implies

$$
\begin{aligned}
\| r(t)\, \Phi(t) \| &\leq (1-\nu(t)) \ \| y(\alpha(t)) \| + \nu(t) \ \| r(\alpha(t))\, \Phi(\alpha(t)) \| \\
&\leq \max\{\| y(\alpha(t)) \|, \| r(\alpha(t))\, \Phi(\alpha(t)) \|\},
\end{aligned}
\tag{18}
$$

being $\nu(t) = \dfrac{r(t)}{r(\alpha(t))}\, Z(t) \leq 1$. Now, for any vector function $v(s)$ and any integer $l \geq 0$, set

$$
|||v|||_l = \sup_{s \in I_l} \| v(s) \| \, .
$$

Therefore, routine calculations and (13) yield

$$
|||r\, \Phi|||_1 = \kappa, \tag{19}
$$

$$
|||r\, \Phi|||_n \leq \max\Big\{|||y|||_{n-1}, |||r\, \Phi|||_{n-1}\Big\}, \qquad n \geq 2. \tag{20}
$$

Now, for all $n \geq 1$ define $\alpha_n = \max\{|||y|||_n, |||r\, \Phi|||_n\}$, so that (15) and (20) imply

$$
|||y|||_n \leq \max\Big\{|||y|||_{n-1}, |||r\, \Phi|||_n\Big\}, \qquad n \geq 1, \tag{21}
$$

and $\alpha_n \ \leq \ \alpha_{n-1}$ for $n \geq 2$. Therefore,

$$
|||y|||_n \leq \alpha_1 \qquad \forall n \geq 2.
$$

Finally, consider the case $n = 1$, for which (15), (16) and (13) yield

$$
|||y|||_1 \leq \max\{\| g(t_0) \|, \kappa\}.
$$

Since $\alpha_1 = \max\Big\{|||y|||_1,\ |||r\, \Phi|||_1\Big\}$, the last inequality provides

$$
\alpha_1 \leq \max\{\| g(t_0) \|, \kappa\}
$$

and then (12) is proved. $\square$

Remark 3 *When G does not depend on $y'(\alpha(t))$, conditions (11) and (12) reduce to*

$$r(t)\,X(t) \leq 1, \qquad \forall t \geq t_0,$$

and

$$\| \, y(t) \, \| \leq \max \left\{ \| \, g(t_0) \, \|, \sup_{t_0 \leq s \leq \xi_1} \| \, r(s) \, G\Big(s, g(\alpha(s))\Big) \, \| \right\}$$

Remark 4 *In the fully linear non-autonomous case, that is*

$$\begin{cases} y'(t) = L(t)\,y(t) + M(t)\,y(\alpha(t)) + N(t)\,y'(\alpha(t)), & t \geq t_0, \\ y(t) = g(t), & t \leq t_0, \end{cases} \tag{22}$$

we can choose $Y(t) = \mu[L(t)]$, $\mu[\cdot]$ being the logarithmic norm, $W(t) = 1$, $r(t) = -1/\mu[L(t)]$, $X(t) = \| \, M(t) + N(t)\,L(\alpha(t)) \, \|$ and $Z(t) = \| \, N(t) \, \|$. As a consequence, for the contractivity we require that $\mu[L(t)] < 0$ and

$$\frac{\| \, M(t) + N(t)\,L(\alpha(t)) \, \|}{-\,\mu[L(t)]} \leq 1 - \frac{\mu[L(\alpha(t))]}{\mu[L(t)]} \, \| \, N(t) \, \| \qquad \forall t \geq t_0.$$

By slightly strengthening the assumption (11), we can state a first result on the asymptotic stability of the solutions.

Theorem 5 *Assume that the functions (5), (6), (8) and (9) fulfil the inequalities $r(t) > 0$,*

$$Y(t) \, \leq \, Y_0 \, < \, 0, \qquad \forall \, t \geq t_0, \tag{23}$$

and

$$\begin{aligned} &\nu(t) \, \leq \, \tilde{\zeta} \, < \, 1, && \forall \, t \geq t_0, \\ &X(t)\,r(t) \, \leq \, k \, (1 - \nu(t)) \, , && \forall t \geq t_0, \qquad k < 1, \end{aligned} \tag{24}$$

where $\nu(t) = Z(t) \, \dfrac{r(t)}{r(\alpha(t))}$. Then we have $\lim_{t \to \infty} y(t) \, = \, 0$ for every initial function $g(t)$ and for every delay $\tau(t)$ satisfying the assumptions (H1) and

(H2) $\lim_{t \to +\infty} \alpha(t) = +\infty.$

PROOF. Observe that, by (23), the function $r(t) = -\dfrac{W(t)}{Y(t)}$ is continuous. In analogy to the previous theorem, we proceed by developing a step-by-step analysis over the intervals $\{I_n\}$. By (10), for every $t \in I_n$, we get

$$\begin{aligned} \| y(\xi_n) \| &\le E(\xi_{n-1}, \xi_n) \ \| y(\xi_{n-1}) \| + (1 - E(\xi_{n-1}, \xi_n)) \sup_{\xi_{n-1} \le s \le \xi_n} (r(s) \ \| \Phi(s) \|) \\ &\le c_n \, |||y|||_n + (1 - c_n) \ |||r\, \Phi|||_n \, , \end{aligned} \tag{25}$$

where $c_n = \exp\Big(\int\limits_{\xi_{n-1}}^{\xi_n} Y(s)\, ds\Big)$ and, by (H1), $c_n < 1 \ \forall n$. Furthermore, for the interval I_{n+1}, (10) yields

$$|||y|||_{n+1} \le \max\Big\{ \| y(\xi_n) \|, |||r\, \Phi|||_{n+1} \Big\}. \tag{26}$$

In turn, by (25), inequality (26) provides

$$|||y|||_{n+1} \le \max\Big\{ c_n \, |||y|||_n + (1 - c_n) \ |||r\, \Phi|||_n \, , |||r\, \Phi|||_{n+1} \Big\}. \tag{27}$$

Now consider (3) and (7). By the assumption (24), we have

$$\begin{aligned} |||r\, \Phi|||_{n+1} &\le \max_{\xi_n \le s \le \xi_{n+1}} (\nu(s) \, |||r\, \Phi|||_n + k \ (1 - \nu(s)) \ |||y|||_n) \\ &= \zeta_{n+1} \, |||r\, \Phi|||_n + k \ (1 - \zeta_{n+1}) \ |||y|||_n \, , \qquad n \ge 2, \end{aligned} \tag{28}$$

where $\zeta_{n+1} = \nu(s_{n+1})$, with $s_{n+1} \in [\xi_n, \xi_{n+1}]$ suitable point. For the sake of conciseness, we omit the rest of the proof, which is completely analogous to that given in Theorem 3.2 of [2] for the linear case, and is based on a suitable analysis of relations (27) and (28). We remark that the hypothesis (H2) is necessary. In fact, if it did not hold, there would exist only a finite number of breaking points ξ_n and we could not conclude with asymptotic stability. $\square$

The following corollary is obtained by assuming that the delay function is bounded from above.

Corollary 6 *Assume the hypotheses of Theorem 5 and that*

(H3) $\tau(t) \le \tau_M, \qquad \forall t \ge t_0, \qquad$ (fading memory assumption).

Then the solution $y(t)$ has an exponential asymptotic decay.

The proof is analogous to that given in [2], relevant to the linear case.

Remark 7 *When G does not depend on $y'(\alpha(t))$, (24) yield the condition $X(t)\, r(t) \leq k < 1$.*

4 The numerical scheme

Given a general ODE with forcing term $\Phi(t)$ of the form

$$y'(t) = f\left(t,\, y(t),\, \Phi(t)\right), \tag{29}$$

with initial condition $y(t_0) = y_0$ and a mesh $\Delta = \{t_0 < t_1 < \cdots < t_n < t_{n+1} < \cdots\}$, let us consider the s-stage RK method

$$\begin{cases} Y^i_{n+1} &= y_n + h_{n+1} \displaystyle\sum_{j=1}^{s} a_{ij}\, K^j_{n+1}, \qquad\qquad i = 1, \ldots, s, \\ K^i_{n+1} &= f\Big(t^i_{n+1}, Y^i_{n+1}, \Phi(t^i_{n+1})\Big), \\ y_{n+1} &= y_n + h_{n+1} \displaystyle\sum_{i=1}^{s} w_i\, K^i_{n+1}, \end{cases} \tag{30}$$

where $t^j_{n+1} = t_n + c_j h_{n+1},\ c_i = \displaystyle\sum_{j=1}^{s} a_{ij},\ i = 1, \ldots, s,$ and $h_{n+1} = t_{n+1} - t_n$. A continuous extension of the s-stage RK method (30) may be obtained by considering the following interpolation:

$$\eta(t_n + \theta h_{n+1}) = y_n + h_{n+1} \sum_{i=1}^{s} w_i(\theta)\, K^i_{n+1}, \qquad 0 \leq \theta \leq 1, \tag{31}$$

where $w_i(\theta)$, $i = 1, \cdots, s$, are polynomials of given degree d such that $w_i(0) = 0$ and $w_i(1) = w_i$. Henceforth we shall assume that $c_i \in [0, 1]\ \forall\, i$. In fact, if $c_i > 1$ for some i, there could be some problems with the considered method of steps, due to the possibility that $\eta(\alpha(t^i_n))$ could be not available from previous computations. A special instance, which will be referred to in the next sections, consists of linear interpolation. In this case we have $w_i(\theta) = \theta\, w_i$ and hence

$$\eta(t_n + \theta h_{n+1}) = (1 - \theta)\, y_n + \theta\, y_{n+1}, \qquad 0 \leq \theta \leq 1. \tag{32}$$

Now we propose a procedure for solving the test problem (1), which acts recursively on the intervals $I_j = [\xi_{j-1}, \xi_j]$, where the true solution $y(t)$ is regular. The procedure, also called *method of steps*, is summarized by the following scheme. On each interval I_j, $j \geq 1$, use a continuous RK method on the mesh $\Delta_j = \{t_{j,0} \equiv \xi_{j-1} < t_{j,1} < \cdots < t_{j,N_j} \equiv \xi_j\}$ in I_j to solve the nonlinear system

$$\begin{cases} y'(t) = F\left(t, y(t), \overline{\Phi}(t)\right), \\ y(\xi_{j-1}) = \eta(\xi_{j-1}), \end{cases} \tag{33}$$

where $\eta(t)$ is the continuous approximation already computed by the numerical method in the previous intervals and the function $\overline{\Phi}(t)$ is given by one of the following schemes.

First scheme: *direct evaluation* (DE):

$$\overline{\Phi}(t) = \begin{cases} G\Big(t, g\big(\alpha(t)\big), g'\big(\alpha(t)\big)\Big), & t \in [t_0, \xi_1], \\ H\Big(t, \eta(\alpha(t)), \overline{\Phi}(\alpha(t))\Big), & t \in [\xi_{j-1}, \xi_j], \quad j \geq 2. \end{cases} \tag{34}$$

Second scheme: *piecewise polynomial approximation* (PA):

$$\overline{\Phi}(t) = \begin{cases} G\Big(t, g\big(\alpha(t)\big), g'\big(\alpha(t)\big)\Big), & t \in [t_0, \xi_1], \\ H\Big(t, \eta(\alpha(t)), \mathcal{Q}^d_{j-1}\left(\overline{\Phi}(\,\cdot\,)\right)\big(\alpha(t)\big)\Big), & t \in [\xi_{j-1}, \xi_j], \quad j \geq 2, \end{cases}$$

$$\mathcal{Q}^d_l : C\left([\xi_{l-1}, \xi_l], \mathbb{C}^m\right) \longrightarrow \Pi^d_l\left([\xi_{l-1}, \xi_l], \mathbb{C}^m\right),$$

where d is the degree of the approximating polynomial and $\mathcal{Q}^d_l$ denotes a linear projector from the space of continuous vector functions $C\left([\xi_{l-1}, \xi_l], \mathbb{C}^m\right)$ into the space of piecewise vector polynomials $\Pi^d_l\left([\xi_{l-1}, \xi_l], \mathbb{C}^m\right)$. To be more precise, $\mathcal{Q}^d_l$ provides a vector polynomial $\overline{\Phi}$ of degree d on every subinterval $[t_{l,k}, t_{l,k+1}]$ defined by the mesh Δ_l.

In the same way as in [2], we can prove the following result on the order of the proposed numerical schemes for the solution of (2)-(3).

Theorem 8 *For the solution of the NDDE (1), consider the DE-scheme, or the PA-scheme, with piecewise polynomial approximation of order p. Assume that the functions F and G are sufficiently smooth. Moreover, suppose to use a continuous RK method of uniform global order p for the numerical solution of (33) on each interval I_n. Then the numerical solution has uniform global order p.*

5 Stability properties of the DE- and PA-scheme

In this section we analyze the contractivity and asymptotic stability properties of the DE- and PA-scheme proposed in Section 4.

Being the proposed numerical schemes based on the recursive solution of ordinary differential systems such as (33), we are interested in determining RK methods which are contractive with respect to the test equation

$$\begin{cases} y'(t) = F\Big(t, y(t), \Phi(t)\Big), \\ y(t_n) = y_n. \end{cases} \tag{35}$$

To this purpose, we recall the following definition from [6].

Definition 9 *A continuous RK method is said to be* BN_f*-stable if the continuous numerical solution* $\eta(t)$ *of (35) satisfies*

$$\max_{0\le\theta\le1} \| \eta(t_n+\theta h) \| \le \max\left\{ \| y_n \|, \max_{1\le j\le s} \| r(t_n+c_j\,h)\,\Phi(t_n+c_j\,h) \| \right\}, \tag{36}$$

for any stepsize $h>0$, $r(t)$ *being the function considered in Lemma 1.*

It has been proven ([6]) that Backward Euler ($p=1$) and 2-stage Lobatto III-C with linear interpolation ($p=2$) are BN_f-stable. These methods are given by the following Butcher tableaux:

$$\begin{array}{c|c} 1 & 1 \\ \hline & 1 \end{array} \qquad\qquad \begin{array}{c|cc} 0 & \frac{1}{2} & -\frac{1}{2} \\ 1 & \frac{1}{2} & \frac{1}{2} \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}.$$

As in Section 3, we shall assume the function $\overline{\Phi}(t)$ continuous in the closed interval $[\xi_{n-1}, \xi_n]$, $n\ge 1$, by considering

$$\overline{\Phi}(\xi_n) = \lim_{t\to\xi_n^-} \overline{\Phi}(t),$$

which exists. The first result we are able to prove concerns the DE-scheme.

Theorem 10 *Consider the DE-scheme for the numerical solution of (1), and assume to use a* BN_f*-stable continuous RK method for the solution of (33). If the hypotheses of Theorem 2 hold, then for any mesh* Δ *the numerical solution* η *satisfies the contractivity property*

$$\| \eta(t) \| \leq \max\{\| g(t_0) \|, \kappa\}, \qquad \forall t \geq t_0.$$

PROOF. The proof is analogous to the one for the stability of the true solutions and presents several similarities to the proof of Theorem 5.1 in [2]. In fact, it is based on the property established by Definition 9 for the RK method and on the assumption that we compute the function $\overline{\Phi}$ "exactly" (that is "recursively" from the previous steps). The first assumption allows to state an inequality for the numerical solution $\eta(t)$ analogous to (21) on each interval I_n, that is

$$|||\eta|||_n \leq \max\left\{|||\eta|||_{n-1}, |||r\,\overline{\Phi}|||_n\right\}. \tag{37}$$

In fact, denoting the mesh relevant to the interval I_n by

$$t_{n,0} \equiv \xi_{n-1} < t_{n,1} < \cdots < t_{n,N_n} \equiv \xi_n \tag{38}$$

and the corresponding stepsizes by $h_{n,k} = t_{n,k} - t_{n,k-1},\ k \geq 1$, (36) yields

$$\max_{0\leq\theta\leq 1} \| \eta(t_{n,0} + \theta h_{n,1}) \| \leq \max\left\{ \| \eta(t_{n,0}) \|, |||r\,\overline{\Phi}|||_n \right\}.$$

By induction on the points $t_{n,k},\ \ k = 1, \cdots, N_n - 1$, we easily get

$$\max_{0\leq\theta\leq 1} \| \eta(t_{n,k} + \theta h_{n,k+1}) \| \leq \max\left\{ \| \eta(t_{n,0}) \|, |||r\,\overline{\Phi}|||_n \right\},$$

and hence (37). By definition of $\overline{\Phi}$, the assumption (11) provides the inequalities

$$|||\overline{\Phi}|||_1 \leq \kappa \quad \text{and} \quad |||\overline{\Phi}|||_n \leq \max\left\{|||\eta|||_{n-1}, |||r\,\overline{\Phi}|||_{n-1}\right\}, \qquad n \geq 2.$$

From here on, the proof of contractivity is completely analogous to the one given in Theorem 2. $\square$

The second result concerns instead the PA-scheme, where piecewise constant or piecewise linear interpolation is used for approximating the function $\overline{\Phi}$. If $t \in [\xi_{n-1}, \xi_n]$, there exists k such that $\alpha(t) \in [t_{n-1,k}, t_{n-1,k+1}] \subseteq [\xi_{n-2}, \xi_{n-1}]$ and therefore

$$\mathcal{Q}^0_{n-1}\left(\overline{\Phi}\big(\,\cdot\,\big)\right)\big(\alpha(t)\big) = \overline{\Phi}(t_{n-1,k}),$$

$$\mathcal{Q}^1_{n-1}\left(\overline{\Phi}\big(\,\cdot\,\big)\right)\big(\alpha(t)\big) = \frac{t_{n-1,k+1}-\alpha(t)}{h_{n-1,k+1}}\,\overline{\Phi}(t_{n-1,k}) + \frac{\alpha(t)-t_{n-1,k}}{h_{n-1,k+1}}\,\overline{\Phi}(t_{n-1,k+1}).$$

Theorem 11 *Consider the PA-scheme with constant or linear interpolant for the numerical solution of (1) and assume to use a* BN_f*-stable continuous RK method for the solution of (33). If the hypotheses of Theorem 2 hold, then for every mesh* Δ *the numerical solution* η *satisfies the contractivity property*

$$\|\,\eta(t)\,\| \le \max\left\{\|\,g(t_0)\,\|,\,\kappa\right\}, \qquad\qquad \forall t \ge t_0.$$

PROOF. The proof proceeds as for the DE-case (Theorem 10), where the bound for $|||\overline{\Phi}|||_n$ remains the same because the piecewise constant or linear interpolation of $\overline{\Phi}$ is still bounded by the maximum value of $\|\,\overline{\Phi}\,\|$. Therefore, also in this case we can assert the inequality (37) and then proceed as in the previous case. $\square$

Now we pass to consider the asymptotic stability of the numerical schemes.

Lemma 12 *[20]*
Let the continuous RK method (30)-(31) be BN_f*-stable and such that* $c_i \in [0,1]\ \forall\, i$*. Then, for any mesh* Δ*, under the condition* $Y(t) \le Y_0 < 0$*, the numerical solution* $\{y_\ell\}$ *of (35) satisfies*

$$\|\,y_{\ell+1}\,\| \le \psi_{\ell+1}\ \|\,y_\ell\,\| + (1-\psi_{\ell+1})\max_{1\le i\le s}\|\,r(t_\ell + c_i\,h_{\ell+1})\,\Phi(t_\ell + c_i\,h_{\ell+1})\,\|,$$

where $0 \le \psi_{\ell+1} \le \Theta(h_{\ell+1})$*,* $\Theta(h)$ *being the error growth function of the RK-method, which depends on* Y_0*.*

Hairer and Zennaro [9] proved that the error growth function $\Theta(h)$ is an asymptotically negative super-exponential function, that is

$$\begin{cases} \Theta(0) = 1, & \\ \Theta(h) < 1, & \forall\, h > 0, \\ \Theta(h')\,\Theta(h'') \le \Theta(h'+h''), & \forall\, h',\ h'' \ge 0, \\ \lim\limits_{h\to\infty}\Theta(h) < 1. & \end{cases} \tag{39}$$

Theorem 13 *Consider the DE-scheme for the numerical solution of (1), and assume to use a* BN_f*-stable continuous RK method for the solution of (33), such that* $c_i \in [0,1]\ \forall\, i$*. If the hypotheses of Theorem 5 hold, then, for any mesh* Δ*, the numerical solution* η *asymptotically vanishes. Furthermore, if condition (H3) holds, the decay of* η *is exponential.*

PROOF. With the previously introduced notation (38), as a consequence of the assumptions and by means of Lemma 12, we get

$$\| \eta(t_{n,1}) \| \leq \psi_{n,1} \ \| \eta(t_{n,0}) \| + (1-\psi_{n,1}) \max_{0\leq\theta\leq 1} \| r(t_{n,0}+\theta h_{n,1})\, \overline{\Phi}(t_{n,0}+\theta h_{n,1}) \|,$$

where

$$\psi_{l,k} \leq \| \Theta(h_{l,k}) \|, \qquad\qquad l=1,2,\cdots, \quad k=1,\cdots,N_l.$$

With standard manipulations, by (38), we arrive at

$$\begin{aligned} \| \eta(\xi_n) \| &\leq (1-\Psi_n) \max_{0\leq\theta\leq 1} \| r\,(\xi_{n-1}+\theta\,\mathcal{H}_n)\ \overline{\Phi}\,(\xi_{n-1}+\theta\,\mathcal{H}_n) \| + \Psi_n \ \| \eta(\xi_{n-1}) \| \\ &\leq \Psi_n\, |||\eta|||_n + (1-\Psi_n)\ |||r\,\overline{\Phi}|||_n\,, \end{aligned} \tag{40}$$

where $\mathcal{H}_n = \sum_{i=1}^{N_n} h_{n,i} = \xi_n - \xi_{n-1}$ and $\Psi_n = \prod_{k=1}^{N_n} \psi_{n,k}$. By Lemma 12, we get

$$\Psi_n \ \leq\ \Theta(h_{n,1})\,\Theta(h_{n,2})\cdots\Theta(h_{n,N_n}) \leq \Theta(\mathcal{H}_n) \ \leq\ \Theta(\tau_0) \ <\ 1,$$

since $\mathcal{H}_n = \xi_n - \xi_{n-1} \geq \tau_0$ and $\Theta(h)$ is non-increasing. We remark that formula (40) plays, in the numerical case, the role of (25). The method being BN_f-stable, for the interval I_{n+1} we get

$$|||\eta|||_{n+1} \leq \max\Big\{\| \eta(\xi_n) \|, |||r\,\overline{\Phi}|||_{n+1}\Big\}. \tag{41}$$

Hence, by applying the estimate (40), inequality (41) yields

$$|||\eta|||_{n+1} \leq \max\Big\{\Psi_n\, |||\eta|||_n + (1-\Psi_n)\ |||r\,\overline{\Phi}|||_n\,, |||r\,\overline{\Phi}|||_{n+1}\Big\}. \tag{42}$$

By the assumption (24) and by definition of DE-scheme, similarly to (28) we have

$$|||r\,\overline{\Phi}|||_{n+1} \leq \bar{\zeta}_{n+1}\, |||r\,\overline{\Phi}|||_n + k\, \Big(1-\bar{\zeta}_{n+1}\Big)\ |||\eta|||_n\,, \tag{43}$$

where $\bar{\zeta}_{n+1} = \nu(\bar{s}_{n+1})$ with $\bar{s}_{n+1} \in [\xi_n, \xi_{n+1}]$ suitable point. Formulæ (41), (42) and (43) play, in the numerical case, the role of formulæ (26), (27) and (28). Now, by virtue of this correspondence, the proof of the theorem proceeds in perfect analogy to that of Theorem 5 and Corollary 6. □

Similarly, relevant to the PA-scheme, we obtain the following result.

Theorem 14 *Consider the PA-scheme with constant or linear interpolant for the numerical solution of (1) and assume to use a* BN_f*-stable continuous RK method for the solution of (33). If the hypotheses of Theorem 5 hold, then, for every mesh* Δ*, the numerical solution* η *asymptotically vanishes. Furthermore, if condition (H3) holds, the decay of* η *is exponential.*

We conclude the paper by giving an algorithmic description of both the classical method (see Section 2) and the new method (based on the PA-scheme) for integrating (1).

Scheme 1. (classical method)

(i) In the first interval $I_1 = [t_0, \xi_1]$:

(1) Compute

$$\overline{\Phi}(t_{n,i}) = G\Big(t_{n,i}, g(\alpha(t_{n,i})), g'(\alpha(t_{n,i}))\Big), \qquad i = 1, \cdots, s.$$

(2) Evaluate the stages of the RK method

$$K_i = F\Big(t_{n,i}, Y_i, \overline{\Phi}(t_{n,i})\Big), \qquad i = 1, \cdots, s.$$

(3) Construct the polynomial approximations $\eta(t)$ of $y(t)$ and $\zeta(t)$ (possibly $\eta'(t)$) of $y'(t)$.

(ii) In the subsequent intervals $I_j = [\xi_{j-1}, \xi_j]$, $j \geq 2$:

(1) Compute the values

$$\overline{\Phi}(t_{n,i}) = G\Big(t_{n,i}, \eta(\alpha(t_{n,i})), \zeta(\alpha(t_{n,i}))\Big), \qquad i = 1, \cdots, s.$$

(2) As in (i)-(2).

(3) As in (i)-(3).

Scheme 2. (novel method (PA))

(i) In the first interval $I_1 = [t_0, \xi_1]$:

(1) Compute

$$\overline{\Phi}(t_{n,i}) = G\Big(t_{n,i}, g(\alpha(t_{n,i})), g'(\alpha(t_{n,i}))\Big), \qquad i = 1, \cdots, s.$$

(2) Evaluate the stages of the RK method

$$K_i = F\Big(t_{n,i}, Y_i, \overline{\Phi}(t_{n,i})\Big), \qquad i = 1, \cdots, s,$$

and then the continuous numerical approximation of the solution $\eta(t)$.

(3) Interpolate $\Phi(t)$ over the time points $\{t_{n,i}\}$ to obtain a polynomial approximation $\overline{\Phi}(t)$.

(ii) In the subsequent intervals $I_j = [\xi_{j-1}, \xi_j]$, $j \geq 2$:

(1) Compute the values

$$\overline{\Phi}(t_{n,i}) = H\left(t_{n,i}, \eta(\alpha(t_{n,i})), \overline{\Phi}\Big(\alpha(t_{n,i})\Big)\right), \qquad i = 1, \cdots, s.$$

(2) As in (i)-(2).

(3) As in (i)-(3).

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